STRENGTH OF MATERIAL

BY

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AN ELEMENTARY STUDY

Prepared for the use of Midshipmen at the U. S. Naval Academy

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PREFACE.

This book has been prepared for the use of the Midshipmen at the U. S. Naval Academy and is designed to cover a short course in the subject taken up in the Department of Mathematics and Mechanics preliminary to the work in the Departments of Ordnance and Gunnery and of Steam Engineering at the Academy.

In arranging the subject matter many of the methods introduced by officers previously on duty in the Department of Mathematics have been employed and the endeavor has been to lead the student to the opening point for the professional work carried on by the other Departments.

CONTENTS.

INTRODUCTION.

Before beginning the study of Strength of Material, let us see what has been discovered by experimenting with test pieces of material and note some of the conclusions arrived at from the results obtained in this way.

Experiment shows us that whenever a force acts on a body formed of any substance the dimensions of that body are changed. In mechanics all bodies were assumed rigid and the results obtained under this assumption were true, for mechanics taught us to find the action of one body on another or the force transmitted by one body to another, while strength of material will teach us the effect *in the body itself* of a force acting upon it. Mechanics showed us that by means of a piece of material force could be moved from one point to another, and strength of material will show us that in transmitting the force the substance forming the conveyance suffers some slight *temporary* deformation if the force is within certain limits; that beyond these same limits the substance suffers *permanent* deformation and if the force be great enough will be completely ruptured.

The study of strength of material will include finding the safe limits of a force to be transmitted by any particular piece of material, finding the deformation caused by transmitting any force, and finding the dimensions of a piece of material in order that it may safely transmit any particular force.

With regard to deformation materials differ greatly, for example the force which will double the length of a piece of rubber will not apparently change the length of a piece of steel of the same size, though both of these substances are *elastic,* and each if stretched within limits will return to its original

length when the stretching force is removed. The force which makes an indentation in a piece of putty will scarcely affect a piece of lead, but in this case the indentation made *will remain* in both these substances as they are *plastic.* Obviously, then, we must experiment with the different materials and find out some of their physical properties before proceeding with a mathematical investigation.

The materials used in building are *elastic* and we determine their physical properties by experimenting with small pieces of them in machines made for the purpose.

Take steel, for example; small test pieces of different shapes are tried under different forces. A pull is applied and we find the test pieces stretch; if we apply pressure the test pieces are compressed. Having applied all sorts of forces we compare our results and find that the stretching or compression is always of the same amount if the same value of force *on unit cube* of the steel is used. We also find that up to a certain limiting value of the force the material will always return to its original shape when the force is removed, but if we go beyond this value we find the piece will not return to its original shape; in other words it is *not* perfectly elastic for forces beyond this value.

If we go through the whole list of materials used in building and test each kind in the same way we will find that they will all behave in a similar manner, the difference will lie in the *amount* of deformation for any force and the *value* of the limiting force beyond which they are not perfectly elastic.

Experimenting further we find the value of this limiting force for the different materials and having it we can compare the various substances as to their usefulness under different circumstances.

If we continue to experiment again and again with a single substance we find that we may apply *as often as we like* a

force less than the limiting one we found, and that the piece will always return to its original shape when the force is removed; but when the force used *exceeds* the limiting one found, if by ever so little, and the force is applied and removed often enough the piece will *break,* though the deformation caused by the first application of the force was too small to be measured or even noticed. From this fact we see that we must never use a piece of material which will have to sustain a force which is in the least greater than the limiting value found by experimenting.

Materials differ in other ways. A piece of glass is easily broken by a light blow and is therefore called *brittle* or *fragile;* wrought iron can be twisted and bent into almost any shape without rupture and therefore is called *malleable* or *ductile.*

Material which has been melted and cast into desired shapes cools quicker at some parts than it does at others, thereby setting up within it internal stresses which are irregularly distributed. Such castings can be broken by a comparatively light blow though they can usually withstand a large pressure. These internal stresses can be removed by a process called *annealing,* which consists in heating the body to a red heat and allowing it to cool *slowly,* thus allowing the particles an opportunity to rearrange themselves.

Metals have the peculiarity that if overstrained they harden in the vicinity of the overstrain and this hardening goes on with time. Thus a bar which is sheared off while cold will finally become extremely hard and brittle near the sheared end, as will a plate in the neighborhood of cold-punched rivet holes. To avoid this bore the rivet holes and saw the bar, or if feasible anneal the cold-sheared bar and the plate near the cold-punched rivet holes.

Now in all practical cases we must of course use material

that will not break, but in addition we must have material which will not change its dimensions to any considerable extent under the applied load.

The experiments by which the constants used in this study have been determined were very carefully conducted and are the results of many independent efforts on the parts of many different scientists. In this work the mathematical investigation only will be touched upon, the experimental part being beyond its scope.

STRENGTH OF MATERIAL

CHAPTER I.

Tension and Compression—Stress and Strain.

1. In the study of strength of material we must consider two ways of arranging the different pieces used: first, when there is to be motion between the parts, and second, when the parts are to be relatively at rest. In the first case, force is transmitted from one piece to another and the combination of pieces is called a machine, the study of which involves the principles of dynamics; the second arrangement is called a framed structure, or simply structure, and we must employ the principles of statics in its investigation. In either arrangement, any two parts in contact have a mutual action between the touching surfaces, and the effects produced by this action depend in great measure upon the way in which it is applied. In any case it tends to change the shape or dimensions of the parts involved and if the force is great enough to crush or break them. So for permanence the machine or structure must be strong enough *in each part* to withstand any force to which it may be subjected. If then we can find the greatest force that any particular piece of material can endure without breaking or suffering a permanent change in shape we can be sure of its remaining intact for all forces within that limit. In the first four chapters we will apply, separately, all the different forces to which a piece of material can be subjected, and as any piece in our machine or structure may have more than one of these forces to sustain at any given instant we will, in the fifth chapter, show the effect of the combined action of two or more of them.

2. Let us first investigate the effect on a piece of material of an external force applied to it, and we will choose for our investigation a straight rod of uniform cross-section and will not consider the force of gravity as acting. We will apply to the end of our rod a pull, F (not sufficient to break or permanently change its shape), which, in order that the rod remain stationary, will require an equal pull in the opposite direction at the other end. These forces tend to tear apart the particles of the material, and as the bar remains intact the particles must be in a different condition, relative to each other, from that in which they were before we applied the pull. If instead of a pull we exert a pressure on one end, an

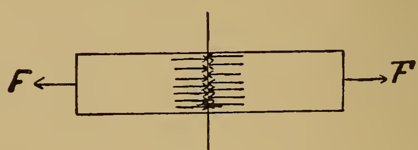

FIG. 1.

equal and opposite pressure must be applied to the other end to keep the bar in equilibrium, and the particles of the material will now tend to crowd together and crush each other. In both of these cases we have arranged our forces so that the bar does not move and they have been taken small enough so as not permanently to change its dimensions. Now as the length of the bar separates the points of application of our forces, there must be some action set up among the particles of the bar itself which transmits the force from one end to the other, or to a common point of action. Let us now imagine a plane passed through the bar perpendicular to its axis: in the first case (the pull, Fig. 1) our forces would tend to pull apart the two pieces of the bar; in the second (not shown) they would press them together, each piece would

tend to move in the direction of the external force acting on it and the amount of this tendency would be equal to that force so that the total action in the bar between the particles on either side of any imaginary section would be equal but opposite to the external forces applied. The action of all the particles on the right side of any section would be equal to *but opposite to* the force on the right end, and of those on the left side equal to *but opposite* to the force on the left end. This must be true in order that the bar remain intact, *i. e.* in equilibrium.

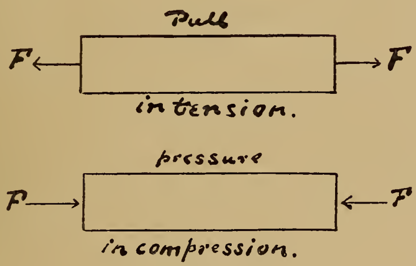

FIG. 2.

3. **Stress.**—When any such action as the above is set up among the particles of a piece of material that piece is said to be under *stress*, and the external force which causes the stress is called the *load*. A rod under the action of a pull is said to be in *tension*, and the pull is the *tensile load*. A rod to which pressure is applied is said to be under *compression*, and the pressure is called the *compressive load*. Hereafter in using the word stress we will mean the amount of action between the particles in *unit cross-sectional area*, and will use "total stress" for the action over the area of the whole cross-section. The stress per unit cross-sectional area is sometimes called "intensity of stress" and is equal to the

external force, F, divided by the cross-sectional area, A. The *forces on one side* of the section *only* are used, so,

$$p \text{ (stress)} = \frac{F}{A}.$$

Of course we would get the *same result* by using the forces on the other side of the section, as the bar must be in equilibrium; they, however, would *act* in the opposite direction.

4. Now let us see what will happen if we gradually increase the load F. We know that all materials are more or less elastic, so as F is increased the bar will stretch or shorten according as F puts it in tension or compression; and up to a certain value of F, the bar will spring back to its original length when F is removed. If we experiment carefully with any material we will find that there is for it a certain value of F, after reaching which, and then having removed F, the bar will be found a little longer or a little shorter than it was originally; in other words, it will have a "permanent set." This value of F for unit sectional area is called the "limiting stress" or "elastic limit" of that material. It is obvious that no part of our machine or structure must be subjected to a force equal to this, for if it be the piece so used is afterward unfit to do its work.

If we go on increasing F the bar will finally break or crush, but at present we are interested only in the elastic limit, the *stress* for which we will call f, and for several materials its value will be found in the table at the end of this chapter.

We know that any stress, p, is equal to $\frac{F}{A}$, so to find the tensile or compressive load any piece of material can sustain unhurt, we put $f = \frac{F}{A}$ or F (the limiting load) $= fA$. This is true for all ordinary lengths of material under tension, and for *short* pieces under compression. We will later (Chapter XI) consider *long* pieces under compression, in which the stress due to bending must be taken into account.

5. In the preceding article we saw that, *within the elastic limit,* a piece of material would, when the load was removed, return to its original length. After voluminous experiments, it has been found that *within the elastic limit,* the extension or compression of a rod under stress varies directly as the stress, or stress is equal to a constant multiplied by the extension or compression. This is known as *Hooke's Law,* from its discoverer. If we let x denote the total extension of a rod of length l, the extension *per unit length* will be $\frac{x}{l}$, and this extension per unit length is called *strain,* and we will denote it by e; then if p denote the stress per *unit area of cross-section,*

$$p = E\frac{x}{l} = Ee,$$

E being a constant found by experiment and called the *modulus of elasticity.* Its value for several materials will be found in the table at the end of this chapter, $E = \frac{p}{e}$.

6. Work Done on a Piece of Material by a Load.—In all cases of overcoming resistance work is done, and we will now find how much work is done on a piece of material on which a load is acting. By Hooke's Law $p = Ee$ so that, as we have equilibrium within the elastic limit, the resistance of the material to the load must equal p per unit area. Before the load is applied the resistance is nothing. Now let us apply the load very slowly. The resistance will increase with the load from zero to the equal of the final load. It follows that the *mean resistance* will be equal to one-half of the final load. Work is equal to force multiplied by the space through which it acts. Due to the load, our rod has stretched through the distance x. Hence the work done is equal to one-half the

final resistance multiplied by x, or if L be the final load the work is equal to $\frac{L}{2} \times x$.

$$W = \frac{Lx}{2}.$$

The following graphic method may be clearer: Let the abscissa represent the extensions, and the ordinates the loads causing them. When $x = 0$, $L = 0$, and as the load increases so will the extension in accord with the formula $p = Ee$, giving us a straight line for the load curve. The work done will now be represented by the area between the load curve, OA, the axis of x, and the ordinate, Ab, representing the final load L.

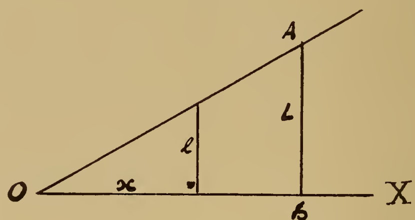

FIG. 3.

$$\text{Work} = \frac{\text{extension}}{2} \times L = \text{area of triangle } OAb.$$

The following is the method by calculus:

$$L \text{ (load)} = pA \text{ (Art. 4)},$$

$$p = E\frac{x}{l} = Ee \text{ (Art. 5)}.$$

The load curve then is

$$L = pA = AE\frac{x}{l} = AEe.$$

The work done by any load L acting through a space dx is

$$Ldx \text{ or } dW = Ldx,$$

but

$$L = EA\frac{x}{l}.$$

$$\therefore\ dW = \frac{EA}{l}\ xdx,$$

which integrated between the limits for x gives

$$W = \frac{EA}{l} \cdot \frac{x^2}{2}\Big]_0^{\text{total extension}}$$

which as $p = \frac{Ex}{l}$ and $L = pA$ gives $W = \frac{Lx}{2}$ for the work as before.

If our bar be stretched to just within the elastic limit we will have

$$f = E\frac{x}{l} \text{ or } x = \frac{fl}{E},$$

also

$$f = \frac{L}{A} \text{ or } L = fA;$$

hence, from the above,

$$W = \frac{f^2Al}{2E} = \frac{f^2}{2E} \times V \text{ (} V \text{ equals volume of bar).}$$

As we have stretched our bar to *just within* the elastic limit, it will return to its original length when we remove the load. This load then does the greatest amount of work that can be done on a piece of material without injuring it. While stretched, the rod has stored in it an amount of energy equal to the work done in stretching it *to its elastic limit.* This stored energy is called the *resilience* of the bar, and the part $\frac{f^2}{2E}$ is the *modulus* of *resilience.*

7. The work done by forces *quickly* applied is much greater than if the force is slowly increased, for, suppose a vertical rod having a collar round its lower end is stretched by a weight W falling from a height h upon the collar. The work done by the falling weight is $W(h + x)$ and this must equal the work done on the rod or $\frac{Lx}{2}$. We have then

$$W(h + x) = \frac{Lx}{2}. \quad \therefore\ L = 2W\left(\frac{h + x}{x}\right),$$

or the slowly applied load which would stretch the rod to the same extent as the falling weight would have to be considerably greater than the weight.

Again, if we imagine a load L to be *instantaneously* applied and to cause a *strain* equal to x and a load L_1, slowly applied which would cause the same strain, the work done in the first case or Lx (force times space) must equal the work done in the second case or $\frac{L_1 x}{2}$. Equating we have $L_1 = 2L$ or a *suddenly* applied load has *twice* the effect of a slowly applied one.

Definitions:

STRESS.—When due to the load on a body, there is mutual action between the particles on either side of a section through the body, so that the particles on one side exert a force on those on the other side, stress is said to exist in that body, and the *intensity of the stress* is the force per unit area of cross-section. Briefly, then, stress is force per unit area of cross-section.

STRAIN.—The ratio of change of length in a body due to the load on it, to the original length; briefly, change of length per unit length.

MODULUS OF ELASTICITY.—The stress which would double the length of a rod provided Hooke's Law held good for that extension; or, it is the ratio of unit stress $\left(\frac{\text{load}}{\text{area}}\right)$ to unit strain $\left(\frac{\text{total extension}}{\text{original length}}\right)$; or ratio of stress to strain within Hooke's Law.

ELASTIC LIMIT.—That stress, which if exceeded will produce a *permanent* change of length; or the maximum stress a material can suffer without being permanently deformed.

RESILIENCE.—When a bar is loaded to its elastic limit, the work done in stretching it is called the resilience of the bar, and the *modulus* of resilience is this work divided by the volume of the bar; or, resilience is the capacity of a body to resist external work.

ULTIMATE STRENGTH is the stress which produces rupture.

WORKING LOAD is the maximum stress to which a piece of material will be subjected in actual practice.

STRENGTH OF MATERIAL.

Material.	Average values in pounds per square inch.					
	Weight per cubic foot. Pounds.	Elastic limit. *f*.	Modulus of elasticity. *E*.	Modulus of elasticity for shear and torsion. *C*.	Ultimate fiber stress.	Limiting shearing stress. *q*.
Steel..........	490	35,000 to 50,000	29,000,000	10,500,000	110,000	50,000
Iron, cast.....	450	6,000 T 20,000 C	15,000,000	5,000,000	35,000	7,850
Iron, wrought	480	25,000	25,000,000	10,000,000	54,000	20,160
Brass, cast....	....	......	9,500,000	....	20,000	4,030
Brass, drawn..	520	......	14.500.000	5,000,000	70,000	......
Copper, cast...	540	......	12,000.000	....	22,000	2,890
Copper, dr'wn	....	......	15,000,000	6,000,000	65,000	......
Stone, granite	160	2,000	6,000,000	1,800,000	2,000	......
Timber........	40	3,000	1,500,000	140,000	10.000	1,200

Examples:

1. A steel bar, 5 ins. long, sectional area $\frac{1}{2}$ sq. in., stretches .007 in. under a load of 20,000 lbs., and shows no permanent elongation when the load is removed. What is the modulus of elasticity of the metal?

Solution:

$$E = \frac{Fl}{Ax} = \frac{20{,}000 \times 5 \times 2}{.007} = 28{,}571{,}428.571 \text{ in.-lbs.}$$

2. A vertical wrought-iron rod 200 ft. long has to lift *suddenly* a weight of 2 tons. What is the area of its cross-section if the greatest *strain* to which wrought iron may be subjected is .0005 for unit length. $E = 30{,}000{,}000$. Neglect the weight of the rod.

Solution: Stress = 8960 lbs. $\frac{F}{A} = p = Ee. \quad \therefore A = \frac{p}{Ee}$

$$= \frac{8960}{30{,}000{,}000 \times .0005} \text{ or } A = .597\tfrac{1}{3} \text{ sq. in.}$$

3. Find the area of cross-section in example 2, taking the weight of the rod into account.

Ans. .611 sq. in., or .625 if rod also is *suddenly* raised.

4. A steel rod $\frac{1}{8}$ sq. in. in sectional area and 5 ft. long is found to have stretched $\frac{3}{100}$ in. under a load of 1 ton. What is the modulus of elasticity of steel.

Ans. 35,840,000 in.-lbs.

5. A chain 30 ft. long and sectional area $\frac{3}{4}$ sq. in. sustains a load of 3900 lbs., an additional load of 900 lbs. is suddenly applied. Find the resilience at the instant the 900 lbs. is applied. $E = 25{,}000{,}000$.

Ans. 25.992 ft.-lbs.

6. A steel rod $3\frac{1}{4}$ ft. long, $2\frac{1}{2}$ sq. ins. sectional area, reaches the elastic limit at 125,000 lbs., with an elongation of .065 in. Find the stress and strain at the elastic limit, the modulus of

elasticity, and the modulus of resilience of steel, and express each in its proper units.

Ans. $f = 50{,}000$ lbs. per sq. in.; $e = .00166\frac{2}{3}$; $E = 30{,}000{,}000$ lbs. per sq. in.; and modulus resilience $= 41\frac{2}{3}$ lbs. per sq. in.

7. A piston rod is 10 ft. long and 7 ins. in diameter. The diameter of the cylinder is 5 ft. 10 ins., and the effective steam pressure is 100 lbs. per sq. in. Find the stress produced and the total alteration in length of the rod for a complete revolution. $E = 30{,}000{,}000$.

Ans. Change in length .0796 in.

8. How much work can be done in stretching a composition rod 5 ft. long and 2 in. in diameter, without injury, if the proof stress of the metal is 2.8 tons per sq. in.? $E = 4928$ ton-ins.

Ans. .15 ton-ins., using $\pi = \frac{22}{7}$.

9. The proof strain of iron being $\frac{1}{1000}$, what is the shortest length of rod $1\frac{1}{2}$ sq. ins. sectional area, which will not take a permanent set if subjected to the shock caused by checking the weight of 36 lbs. dropped through 10 ft., before beginning to strain the rod? $E = 30{,}000{,}000$.

Ans. 192.31 ins.

10. Find the shortest length of steel rod, 2 sq. ins. sectional area, which will just bear, without injury, the shock caused by checking a weight of 60 lbs. which falls through 12 ft. before beginning to strain the rod. $E = 30{,}000{,}000$. Modulus of resilience $= 15$ lb.-ft.

Ans. 24.05 ft.

11. A brass pump rod is 5 ft. long and 4 ins. in diameter and lifts a bucket 28 ins. in diameter, on which is a pressure of 6 lbs. per sq. in., in addition to the atmosphere, against a

vacuum below the bucket which reduces the atmospheric pressure to 2 lbs. What is the stress in the rod and the total extension per stroke? $E = 9{,}000{,}000$.

Ans. Stress 925 lbs. per sq. in., and extension .0061 in.

12. Assuming a chain twice as strong as the round bar of which the links are made, what size chain must be used on a 20-ton crane with three sheaved blocks if $f = 6000$?

Ans. Diameter of section of metal .8899 in.

13. A piston rod is 9 ft. long and 8 ins. in diameter. The diameter of the cylinder is 88 ins. and the effective pressure is 40 lbs. per sq. in. What is the stress produced and the total alteration in length of the rod per revolution? $E = 29{,}000{,}000$.

Ans. $x = .0359$ in.

14. Find the work done in Example 13, and find the resilience of the rod if $f = 12$ tons.

Ans. $W = 3.88$ in.-tons. $R = 52.5$ in.-tons.

15. The stays of a boiler in which the pressure is 245 lbs. per sq. in. are spaced 16 ins. apart. What must be their diameter if the stress allowed is 18,000 lbs. per sq. in.?

Ans. $2\frac{1}{8}$ ins.

16. What is the area of the section of a stone pillar carrying 5 tons if the stress allowed is 150 lbs. per sq. in.?

Ans. 75 sq. ins.

17. What is the length of an iron rod (vertical) which will just carry its own weight? $f = 9000$ lbs. per sq. in.

Ans. 2700 ft.

18. The coefficient of expansion of iron is .0000068 in. per degree F. An iron bar 18 ft. long, $1\frac{1}{2}$ ins. in diameter, is secured at 400° F., between two walls. What is the pull on the walls when the bar has cooled to 300° F.?

Ans. 34,862 lbs.

19. Coefficient of expansion of copper is .0000095 in. $E = 17,000,000$. A bar of iron is secured between two bars of copper of the same length and section at 60° F. What are the stresses in the bars at 200° F.?

Ans. In copper 2958, iron 5916 lbs. per sq. in.

20. A bar of iron is 3 ft. long, 2 ins. in diameter. The middle foot is turned down to one in. diameter. Compare the resilience with the original bar and with a uniform bar of the same weight.

Ans. (1) 1 to 8; (2) 1 to 6.

CHAPTER II.

SHEARING.

8. We will next apply force to produce shearing. In studying shearing we will use the same sort of rod we used in the preceding chapter but will apply our force in a different way. To get the proper effect we will slip our rod through holes bored in two extra pieces of material as in Fig. 4, and then apply opposite and equal pulls to the two extra pieces so that the effect on the rod will be to pull one part up and the other down. If we pull hard enough our rod will be sliced off

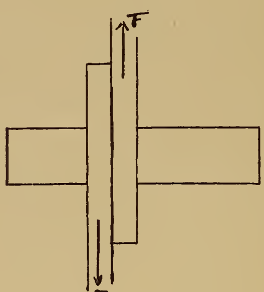

FIG. 4.

smoothly as though a plane had been passed through it perpendicular to its axis. It will be *sheared* off. Practically this is as near as we can approximate to pure shearing. In theory the two forces, F, should act on either side of a section of the rod in parallel planes indefinitely close to each other so that their pull would induce no tendency to turning or bending; then whether we actually sheared the bar through or not, the stress in the bar *on one side* of the section would

be the force, F, divided by the area A, of the section of the bar, or, $q = \frac{F}{A}$.

It will be noticed that the direction of the shearing stress is *parallel* to the section, while in tensile or compressive stress the direction is normal to the section; for this reason shearing is sometimes called *tangential* stress.

9. Two plates bearing a longitudinal load and held together by a riveted joint present a good illustration of shearing. The rivet is under almost pure shear if it closely fits the rivet holes, and the bearing surfaces of the plates are plane. Fig. 5 shows a section and plan of a single-riveted lap joint. The distance between the centers of the rivet holes is called the

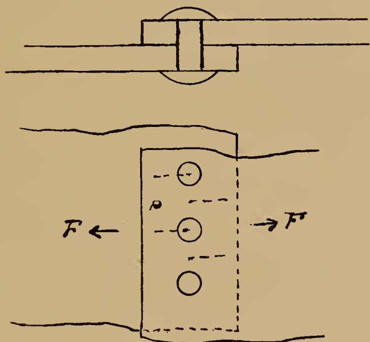

FIG. 5.

pitch, p, and it is obvious that each rivet supports the load on a strip of plate equal in width to the pitch of the rivets. So that if F is the force acting along this strip of plate, the shearing stress, q, on the rivet supporting it is equal to $\frac{F}{A}$ as before, or as A is $\frac{\pi d^2}{4}$, where d is diameter of the rivet, we have for the shearing stress $q = \frac{F}{\frac{\pi d^2}{4}}$. If we put the

limiting shearing stress for the material of which our rivet is made for q, we can find the diameter of the rivet we must use under the conditions, $d = 2\sqrt{\frac{F}{\pi q}}$.

In working joints, such as pin joints, the action is not pure shear, but, due to the clearance necessary for a working fit there is some bending. Experiment has proved that the stress at the center of the pin of such a joint is $\frac{4}{3}$ of that found by the above formula. To find the diameter of the pin necessary for a joint like that in Fig. 6, we first notice that there are two sections of the pin to be sheared,

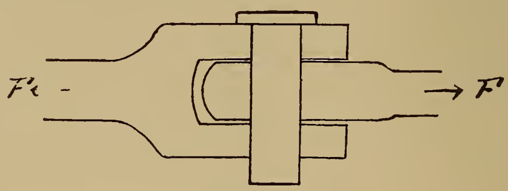

FIG. 6.

then,

$$\left\{\begin{array}{l}\text{shearing stress}\\ \text{at center of pin}\end{array}\right\}\cdots = \frac{4}{3}q = \frac{4}{3}\cdot\frac{F}{2A} = \frac{8F}{3\pi d^2};$$

or,

$$d = \sqrt{\frac{8F}{3\pi q}}.$$

The limiting shearing stress for several materials is given in the table at the end of Chapter I.

10. Usually the value of F for the above is readily found, as the thrust on a piston rod, etc.; but for the force acting on riveted joints the ordinary steam boiler will give us a good example. Fig. 7 is a section through a boiler designed to carry a steam pressure of s pounds per square inch, its radius is r, and we wish to find the tangential force at any point A.

Draw the perpendicular diameters AB and CD, then on a ring 1 inch wide there will be a pressure of s pounds on each inch of the circumference. Now if we should resolve horizontally and vertically the pressure on each one of these square inches, the sum of *all* the horizontal components would be zero. If, however, we take the sum of all the horizontal components on the right half of the ring we will get all the forces which act toward the right; of course those to the left

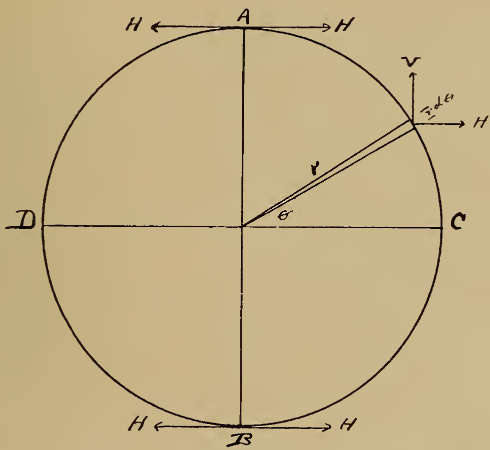

FIG. 7.

of the diameter AB will be equal but opposite in direction. It will be further noticed that to support these forces we have the boiler material at A and also at B and both these points support equal amounts, so we can divide the force acting on the semicircle by 2 to get that part which acts at A or B; *or*, we can take the sum of the horizontal components of the pressure on that part of the ring from C to A which will give the same result. Finding this sum is most readily done by calculus. Take any point on the circumference and let its

angular distance from C be θ, then $rd\theta$ (remembering our strip is 1 inch wide) will be an element of area on which the pressure is $s \times rd\theta$ acting radially and of which the horizontal component is $srd\theta \cos \theta$, and if we integrate this expression between the limits 0 and $\frac{\pi}{2}$ we will get the sum of all these components, or the force H.

$$H = sr \int_0^{\frac{\pi}{2}} \cos\ \theta d\theta = sr \left[\sin\ \theta \right]_0^{\frac{\pi}{2}} = sr.$$

Having H the force on a strip 1 inch wide the force on a strip supported by one of our rivets is found by multiplying this by the pitch, or $F = psr$, so that the shearing stress on the rivet is

$$q = \frac{psr}{A}.$$

The vertical components have no tensile effect at A, but form all the tensile effect at C.

11. In this connection we can find the required thickness of a cylindrical shell, such as a steam pipe, remembering that it is under *tensile* stress not shearing and changing our constants accordingly. If l is the length considered and t the thickness of the plate, the sectional area of the metal will be lt, and if f is the tensile strength allowed, the resistance of the shell will be flt; this must be balanced by the force lF, which from Art. 10 is lsr:

$$\text{so } flt = lsr; \text{ or } t = \frac{sr}{f}.$$

If we wish the thickness of boiler plates, we must, since boilers are built of plates with riveted joints, divide our result by the efficiency of the joint, strength of joint being equal to the strength of the solid metal multiplied by the efficiency of the joint.

12. We have seen how to find the shearing stress on a rivet of a single-riveted lap joint, but we must remember we have taken out of our plate a piece of metal equal to the diameter of the rivet hole, so that we have left in the strip of plate, to bear the whole force F, a section whose area is $(p - d)t$, and the *tensile* stress on this section must not exceed the *tensile* strength of the material, or to balance; $f = \frac{F}{t(p-d)}$; transposing, the pitch for a single-riveted lap joint is

$$p = \frac{F}{ft} + d.$$

There is another kind of joint called the butt joint, where a narrow strip of material, or strap, covers the edges of the plates and is riveted to both. This joint is called a single butt joint, single-riveted (Fig. 8), and is treated exactly as a single-riveted lap joint.

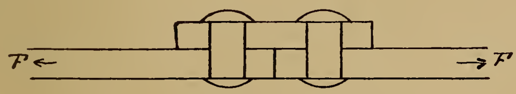

FIG. 8.

If there are two straps used, one on either side of the plates as in Fig. 9, the joint is called a double butt strap joint,

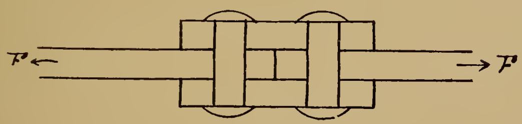

FIG. 9.

single-riveted, and here it will be noticed we have two sections of rivet to shear, so our formula becomes $q = \frac{F}{2A}$.

Now suppose we have two rows of rivets in either a lap joint or single butt joint (where more than one row is used the rivets are generally staggered as in Fig. 10, shown in the plan) and we will have *two* sections of rivets to shear, and our

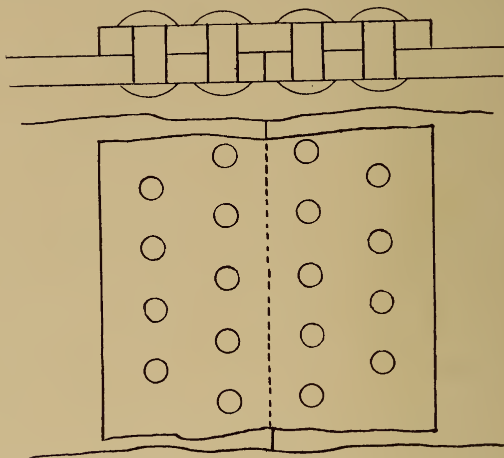

FIG. 10.

formula will be the same as for the single-riveted double butt strap,

$$q = \frac{F}{2A},$$

and if we have for these joints n rows of rivets, the formula becomes

$$q = \frac{F}{nA}.$$

If we use double butt straps we will have double the number of sections to shear, so have to divide the above value of F by 2; or, for double butt straps

$$q = \frac{F}{2nA}.$$

	Lap joint or single butt strap.	Double butt strap.
Single-riveted . .	$q = \frac{F}{\frac{\pi d^2}{4}}$ or $F = \frac{q\pi d^2}{4}$	$q = \frac{F}{2 \cdot \frac{\pi d^2}{4}}$ or $F = \frac{2q\pi d^2}{4}$
Double-riveted .	$q = \frac{F}{2 \frac{\pi d^2}{4}}$ or $F = \frac{2q\pi d^2}{4}$	$q = \dots \frac{F}{4 \cdot \frac{\pi d^2}{4}}$ or $F = q\pi d^2$
n rows of rivets .	$q = \frac{F}{n \frac{\pi d^2}{4}}$ or $F = \frac{nq\pi d^2}{4}$	$q = \frac{F}{2n \frac{\pi d^2}{4}}$ or $F = \frac{2nq\pi d^2}{4}$

13. If we have more than one row of rivets the *plate* will, if it be ruptured, carry away along the *outer* row of rivets *in every case;* for (see Fig. 11) if it did not, there would be one or more rows of rivets to be sheared in addition to the

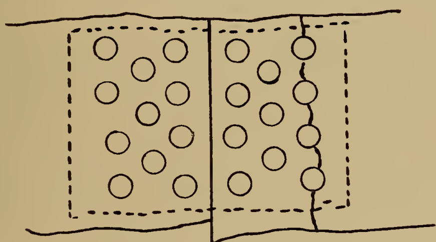

FIG. 11.

carrying away of the plate, and clearly the rupture would occur where it would require the least force. Consequently, as far as the *stress on the plate* is concerned, it may always be computed as we have it in the final part of Art. 12, using the limiting stress allowed for the material; or,

$$f(p-d)t = F.$$

At the end of Art. 12 we have formula giving F for the shearing stress of the material of the rivets and if we equate these two values of F we can find the pitch of our rivets for equal

strength (that is, rupture will be as likely to occur by shearing the rivets as by the plate carrying away), if we know their diameter and the thickness of the plates.

Equating we have

$$\frac{nq\pi d^2}{4} = f(p - d)t,$$

$$\therefore\ p = d + \frac{nq\pi d^2}{4ft},$$

which gives the pitch where particular plates and rivets are to be used.

In these formulæ

$p =$ pitch of rivets,
$d =$ diameter of rivets,
$n =$ number of rows,
$q =$ limiting shearing stress for rivets,
$f =$ limiting tensile stress for plate,
$t =$ thickness of plate.

14. When a bar is subjected either to tension or compression there is shearing stress along *any* oblique section. Let

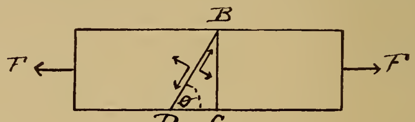

FIG. 12.

the angle $BDC = \theta$, then the *intensity* of the force F (acting in either direction) along the section DB is $\frac{F}{\text{area } DB}$, but the area DB equals the area of $BC \times \operatorname{cosec} \theta$, or if A is the area of a right section, the intensity of the force on DB is $\frac{F}{A \operatorname{cosec} \theta}$. The component of this force resolved along DB is

$\dfrac{F}{A \text{ cosec } \theta} \cos \theta = \dfrac{F}{A} \sin \theta \cos \theta = \dfrac{F}{2A} \sin (2\theta)$. This being a tangential force is shearing stress. $\dfrac{F}{A \text{ cosec } \theta}$ resolved perpendicular to DB is $\dfrac{F}{A \text{ cosec } \theta} \sin \theta = \dfrac{F}{A} \sin^2 \theta$, the normal stress. When $\theta = 0$ or $\frac{\pi}{2}$ the shearing stress is 0, but for any other value of θ it is a finite quantity, which proves our proposition. When $\theta = 0$ the normal stress is 0, but increases with θ to a maximum when $\theta = \frac{\pi}{2}$, which is as it should be.

Examples:

1. Two wrought-iron plates, 3 ins. wide by $\frac{1}{2}$ in. thick are lap-jointed by a single rivet, 1 in. in diameter. What will be the pull required to break the joint? The tensile strength being 18 tons per sq. in. What is the efficiency of the joint?

Solution: Area (plate) $= 3 \times \frac{1}{2} - 1 \times \frac{1}{2} = 1$ sq. in. $\therefore$ Strength $= 18$ tons. Area of section without rivet hole $= \frac{3}{2}$ sq. in. $\therefore$ Strength $= \frac{3}{2} \times 18 = 27$ tons. Efficiency of the joint is $\frac{18}{27}$ or $\frac{2}{3} = 66\frac{2}{3}\%$. Area of rivet section $= \frac{\pi}{4}$. If it is of the same material as the plate the force necessary to shear it is $F = \dfrac{q\pi d^2}{4} = \dfrac{18 \times 22}{4 \times 7} = 14.13 +$ tons and the joint would break by shearing the rivet.

2. A cylindrical vessel with hemispherical ends, diameter 6 ft., is exposed to internal pressure 200 lbs. above the atmosphere. It is constructed of solid steel rings riveted together. If $f = 7$ tons per sq. in., how thick must the metal be, and what is the longitudinal stress in the metal of the ring joint whose section is $\frac{7}{10}$ that of the solid plate?

Solution: $t = \frac{sr}{f} = \frac{200 \times 6 \times 12}{2 \times 7 \times 2240} = .459$ in.

Longitudinal stress $= \pi r^2 p = \frac{7}{10} .2\pi r t q$ or $q = \frac{10\pi r^2 p}{7.2\pi r t} = \frac{5rp}{7t}$

$$q = \frac{5 \times 3 \times 12 \times 200}{7 \times .459 \times 2240} = 5 \text{ tons.}$$

3. What must be the thickness of a copper pipe, $\frac{3}{4}$ in. in diameter, to sustain a pressure of 1350 lbs. per sq. in.? $f = 950$ lbs. per sq. in.

Ans. .533 in.

4. A single-riveted lap joint, plate $\frac{1}{2}$ in. thick, is under a load of 3 tons per sq. in. of plate section. Rivets $\frac{3}{4}$ in. in diameter, pitch $1\frac{7}{8}$ ins. What is the shearing stress on the rivets and the efficiency of the joint?

Ans. Shearing stress 3.82 tons. Efficiency 60%.

5. Two plates, $\frac{3}{4}$ in. thick, are double-riveted with double butt straps, rivets 1 in. in diameter. The shearing strength of the rivets is $\frac{5}{6}$ the tensile strength of the plates. Find the pitch and the efficiency of the joint.

Ans. Pitch $4\frac{1}{2}$ ins. Efficiency .7778.

6. What is the pitch of the rivets in a treble-riveted, double butt strap joint between plates $\frac{1}{2}$ in. thick, rivets $\frac{3}{4}$ in. in diameter, if the tensile stress of the plates is limited to 10,000 lbs. per square in., and the shearing stress of the rivets to 8000 lbs. per sq. in.? What is the efficiency of the joint?

Ans. Pitch $\frac{629}{140}$ in. Efficiency 83 + %.

7. A cylindrical boiler 8 ft. 4 ins. in diameter is under 100 lbs. per sq. in. pressure. What must be the thickness of the plates that the stress may not exceed 4000 lbs. per sq. in.

Ans. $1\frac{1}{4}$ ins.

8. What is the pitch of 1-in. rivets in a double-riveted lap joint between $\frac{1}{2}$-in. steel plates? Tensile stress 13.2 tons and

shearing stress 10.5 tons per sq. in. What is the efficiency of the joint?

Ans. Pitch $3\frac{1}{2}$ ins. Efficiency $71 + \%$.

9. The steel plates of a girder are $\frac{3}{4}$ in. thick, treble-riveted with double butt strap, with 1-in. rivets. Shearing stress of the rivets is $\frac{5}{6}$ the tensile strength of the plate. What is the pitch?

Ans. 6.24 ins.

10. In a pin joint (Fig. 6) the shearing stress of the pin is $\frac{4}{5}$ the tensile stress of the rod. Compare the diameters.

Ans. 1 to 8, nearly.

11. A square bar of steel is under a tensile stress of 4 tons and a compressive stress of 2 tons at right angles to its axis. What are the shearing and normal stresses on a plane making the angle $\tan^{-1}\dfrac{1}{\sqrt{2}}$ with the axis?

Ans. S. $2\sqrt{2}$ tons. $N = 0$.

12. What should be the pitch of 1.25-in. rivets in a treble-riveted, double butt strap joint between 1-in. plates, if the resistance to shearing is $\frac{3}{4}$ the resistance to tearing?

Ans. 6.77 ins.

13. What pressure in a copper pipe, $\frac{4}{10}$ in. in diameter, and .02 in. thick, will stress the copper to 8000 lbs. per sq. in.?

Ans. 800 lbs.

CHAPTER III.

TORSION.

15. In Chapters I and II we have seen the effects of tension or compression, and of shearing. In this chapter we will fix firmly one end of our rod and twist it by applying a turning moment to the free end. This will subject it to torsion, with the effect that any right section will be turned about its center in an amount depending upon the distance of the section from the fixed end.* The stress between the particles on either side of any section will be parallel to the section, or

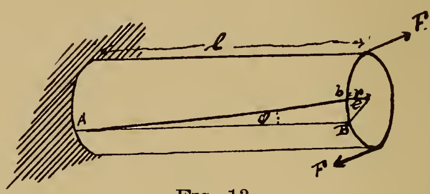

FIG. 13.

it is tangential stress. Clearly, then, torsion is a kind of shear. If before applying our turning moments, we draw a line, AB (Fig. 13), on our test piece, parallel to its axis, it will, after the moment is applied, be found to have deformed into a helix or screw thread, and the point B will now be at b. The angle BAb or ϕ, is the angle of torsion, and is proportional to the radius of the rod. The angle at the center, θ, is called the angle of twist, and is proportional to the length

* We might have applied a turning moment to both ends in opposite directions, in which case the section at the middle of the rod (the fixed end as taken above) would have remained stationary, though it would have been stressed as the others.

of the rod. The distance Bb is equal to $r\theta$, but it is also equal to $l\phi$; as ϕ is a very small angle, so that AB and Ab are practically equal,

$$r\theta = l\phi; \text{ or } \phi = \frac{r\theta}{l}. \tag{1}$$

We have, by experiment, the equation $q = C\phi$ for torsion, where q is the stress at the surface and C is the *modulus* of distortion or torsion, just as we have $p = Ee$ to hold in tension or compression. ϕ, the angle of torsion being a measure of the *strain*. We have then two values of ϕ,

$$\phi = \frac{r\theta}{l}, \text{ and } \phi = \frac{q}{C}; \therefore \frac{r\theta}{l} = \frac{q}{C}; \text{ or } \theta = \frac{ql}{Cr}. \tag{2}$$

It is obvious that for any material, θ, on any section, will be constant for any stress. This being true, our equation for θ proves that q varies with r, or if q, q_1, q_2, be the stress on the surface of rods of the same length of radii r, r_1, r_2, then

$$\frac{q}{r} = \frac{q_1}{r_1} = \frac{q_2}{r_2}. \tag{3}$$

This is also true by experiment, for radii drawn on any section remain straight lines after torsion.

16. Let us find the stress in a tube so thin that the intensity of the stress due to torsion will be practically the same over the whole sectional area. If we call this intensity of stress q, the mean radius of the tube r, and t the thickness, then the area of any section will be $2\pi rt$, and the stress on this area will be $q.2\pi rt$, and the moment of this stress about the center of the section will be $q.2\pi rt.r$. This must balance the moment of the force applied to twist the tube, and calling this moment T:

$$T = 2\pi r^2 tq; \text{ or, } q = \frac{T}{2\pi r^2 t}. \tag{1}$$

Now putting this value of q in equation (2) of Art. 15, which is true for either hollow tubes or solid rods, we have

$$\theta = \frac{lT}{2C\pi r^3 t} \qquad (2)$$

for the angle of twist of a thin tube. (Values of C at end of Chapter I.)

17. Let us now take a tube whose thickness must be considered. Let r_1 be its external and r_2 its internal radius (Fig. 14), and let q equal the stress at a distance r from the

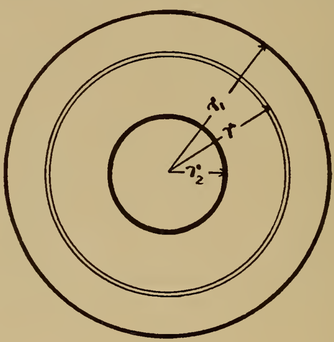

FIG. 14.

center of the tube. Call q_1 the maximum stress which is at the surface, then from equation (3), Art. 15,

$$\frac{q}{r} = \frac{q_1}{r_1}, \text{ or } q = \frac{q_1 r}{r_1}.$$

The element of area at a distance r from the center is $2\pi r dr$, the stress on it is $q.2\pi r dr$, but from above $q = \frac{q_1 r}{r_1}$, so in terms of the maximum stress the stress on the element is $\frac{q_1 r}{r_1}.2\pi r dr$ and the moment of this stress is $\frac{q_1 r}{r_1}.2\pi r^2 dr$ or $\frac{2\pi q_1}{r_1} r^3 dr$, which if integrated between the limits of r which

are r_2 and r_1 will give us the moment of the resistance offered by the tube to the twisting moment applied, or

$$T = \frac{2\pi q_1}{r_1} \int_{r_2}^{r_1} r^3 dr = \frac{2\pi q_1}{4r_1}\left(r_1^{\,4} - r_2^{\,4}\right).$$

Knowing T this gives us $q_1 = \dfrac{2r_1 T}{\pi\,(r_1^{\,4} - r_2^{\,4})}$

if now $r_2 = 0$; or in other words, if the tube is solid,

$$q_1 = \frac{2T}{\pi r_1^{\,3}}.$$

18. It is clear that if we give to q_1 the limiting value for the material, we can find the diameter of the solid rod that will carry a given twisting moment. This formula is used to design shafts to transmit a given horsepower. Let a be the length of the crank arm, and F the mean force acting on it during a revolution; then the mean twisting moment is equal to aF, and the work done per revolution is equal to $aF.2\pi$ (care must be taken to use the same units throughout). The energy of the machine per revolution is $\dfrac{\text{H. P.} \times 33{,}000}{N}$ foot-pounds, where H. P. is the indicated horsepower and N the number of revolutions per minute. This is also the work done per revolution, and if divided by 2π will give the *mean* twisting moment. The *mean* twisting moment is less than the maximum twisting moment, but the shaft must be designed to carry the greatest, which is equal to a constant times the mean, or $T = KT_m$. We put the greatest moment equal to $\dfrac{q\pi r^3}{2}$, substitute the limiting stress allowed for q and solve for r. We can get the *greatest* twisting moment if we take the force acting on the piston, through the piston and connecting rods to the end of the crank, *when the crank is perpendicular to the connecting rod,* and multiply this by the length of the crank, remembering that the force acting along

the connecting rod is the force acting on the piston multiplied by the secant of the angle the piston rod makes with the connecting rod.

19. To compare solid and hollow shafts, let the radii of the hollow shaft be r_1 and r_2, and that of the solid shaft be r; then if T is the resistance offered by the solid shaft and T_1 that by the hollow shaft

$$\frac{T_1}{T} = \frac{\frac{\pi q}{2r_1}(r_1^4 - r_2^4)}{\frac{\pi q r^3}{2}} = \frac{r_1^4 - r_2^4}{r_1 r^3},$$

which gives us the ratio for shafts of the same material. Now if the sectional areas are the same $\pi r^2 = \pi(r_1^2 - r_2^2)$ or $r = \sqrt{r_1^2 - r_2^2}$, substituting

$$\frac{T_1}{T} = \frac{1 + \frac{r_2^2}{r_1^2}}{\sqrt{1 - \frac{r_2^2}{r_1^2}}},$$

which shows that the nearer r_2 approaches r_1, or, the thinner the metal of the hollow shaft becomes, always *retaining the same sectional area* as the solid, the nearer the ratio of strength of hollow to strength of solid would approach ∞. There is, however, a limit to the thinness of our shaft, as it must be able to support its own weight without buckling. If we assume that the ratio of the radii of the hollow shaft is as 2 to 1, and that the sectional areas are equal, we get $\frac{T_1}{T} = 1.44$, or the hollow shaft is nearly half again as strong as the solid one under these conditions.

20. Substitute in equation (2) of Art. 15 the value of q found in Art. 17 for the solid shaft, and we get the angle of twist

$$\theta = \frac{2Tl}{\pi C r^4}$$

for a solid shaft. If we use q found in that article, for the hollow shaft we get

$$\theta = \frac{2Tl}{\pi C(r_1^4 - r_2^4)}$$

for the hollow shaft. Now if T is known and θ can be measured, we can find the value of C; or, if C is known and θ measured we can find T, and thence the horsepower that a rotating shaft is transmitting.

Examples:

1. What must be the diameter of a shaft to transmit a twisting moment of 352 ton-ins., the stress allowed being $3\frac{1}{2}$ tons per sq. in.? What H. P. would this shaft transmit at 108 revolutions per minute, the maximum twisting moment being $\frac{55}{36}$ of the mean?

Solution:

$$q = \frac{2T}{\pi r^3} \text{ or } r^3 = \frac{2T}{\pi q} = \frac{2 \times 352 \times 7 \times 2}{22 \times 7} = 64.$$

$\therefore$ $r = 4$ ins.; $d = 8$ ins.

$$T = \frac{55}{36} \cdot \frac{\text{H. P.} \times 33{,}000 \times 12}{\text{N} \times 2\pi \times 2240}.$$

$$\therefore \text{ H. P.} = \frac{352 \times 2240 \times 108 \times 2 \times 22 \times 36}{33{,}000 \times 12 \times 7 \times 55} = 884.736.$$

2. Find the size of a hollow shaft to replace the preceding, exterior diameter to be $\frac{8}{5}$ the interior. What is the weight of metal saved in a steel shaft 60 ft. long?

Ans. Exterior diameter 8.454 ins. Interior diameter 5.284. Weight saved 3213 lbs.

3. Find the diameter of a solid shaft to transmit 9000 H. P. at 140 revolutions per minute. The stress allowed being 10,000 lbs. per sq. in., and the maximum twisting moment $\frac{3}{2}$ the mean.

Ans. $d = 14.5695$ ins.

4. Find the size of a hollow steel shaft to replace the above, internal diameter being $\frac{9}{16}$ of the external. What is the saving in weight for 60 ft. of shafting?

Ans. External diameter = 15.091 ins. Internal diameter = 8.4886 ins. Weight saved 8893.5 lbs.

5. Compare the strength of a solid wrought-iron shaft with a hollow steel shaft of the same external diameter. The internal diameter of the steel shaft being $\frac{1}{2}$, the external and elastic strength of steel $\frac{3}{2}$ that of iron.

Ans. As 32 is to 45.

6. A solid shaft fits exactly inside a hollow shaft of equal length. They contain the same amount of material. Compare their strengths when used separately.

Ans. As 3 is to $\sqrt{2}$.

7. The resistance of a twin-screw vessel at 18 knots is 44,000 lbs. At 95 revolutions per minute what will be the twisting moment on each shaft? What is the H. P.?

Ans. $T = 360$ in.-tons. H. P. = 2432.

8. The pitch of a screw propeller is 14 ft.; the twisting moment on the shaft is 120 ton-ins.; the mean diameter of the thrust bearing rings is 15 ins.; coefficient of friction is .05; find the thrust and the efficiency of the thrust bearing.

Ans. Thrust = $4\frac{1}{2}$ tons. Efficiency of bearing $98\frac{3}{5}$ %.

9. The angle of torsion of a shaft is not to exceed 1° for each 10 ft. of length; what must be its diameter for a twisting moment of 16,940 ft.-lbs.

$$\left[C = 10{,}976{,}000 \text{ in.-lb. units.} \quad \pi = \frac{22}{7}\right].$$

Ans. Diameter = 6 ins.

10. A solid steel shaft, 10.63 ins. in diameter, is transmitting 12,000 H. P. at 200 revolutions per minute. If the maximum twisting moment is $\frac{5}{4}$ the mean, what is the maximum stress (torsion) in the shaft?

Ans. 20,000 lbs. per sq. in.

11. If the modulus of rigidity be 4800 in in.-ton units, what is the greatest stress to which a shaft should be subjected that the angle of torsion may not exceed 1° for each 10 diameters of length?

Ans. 4.2 tons per sq. in.

12. In changing engines in a ship, the number of revolutions is increased $\frac{1}{3}$; H. P. is doubled; the ratio of maximum to mean twisting moment is changed from $\frac{3}{2}$ to $\frac{5}{4}$; and the strength of material of the shaft is 25% greater. What is the relative size of the new shaft?

Ans. The same size.

13. Find the angle of torsion of a steel tube, 6 ft. long, $\frac{1}{8}$ in. thick; mean diameter 12 ins., shearing stress allowed 4 tons per sq. in.

Ans. 5°.7.

CHAPTER IV.

BENDING.

21. There is one more way in which we can introduce stress into our rod and that is by bending it. If we rest our rod on supports at its ends and load it at its middle, it will bend into a curve. The plane of this curve is called the *plane of bending* (*OABO,* Fig. 15). The particles on the concave side of the rod will tend to crowd together and will be in compres-

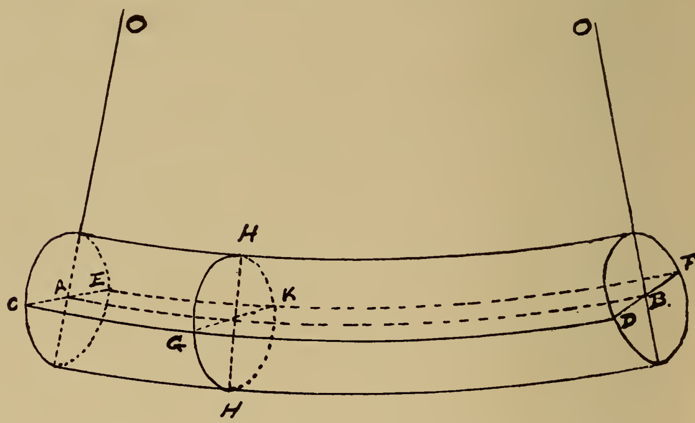

FIG. 15.

sion, while those on the convex side will tend to pull apart and will be in tension; obviously, there will be some intermediate plane, perpendicular to the plane of bending, where there will be neither tension nor compression, a plane of no stress. This plane is called the *neutral surface.* The *neutral line* is the intersection of the neutral surface with the plane of bending,

and it gives us the curve into which the rod bends, and is therefore called *the elastic curve.* The intersection of the neutral surface with any section of our rod perpendicular to its axis is called the *neutral axis* of that section. In Fig. 15, the lines AO and BO meet, O being their point of intersection. AOB is the *plane* of *bending.* $CEFD$ is the *neutral surface.* AB is the *neutral line* or *elastic curve;* and if HH is any section of the rod perpendicular to AB, then GK is its *neutral axis.*

22. Now before bending, all sections of our rod are parallel, but after bending these same sections (assuming that they

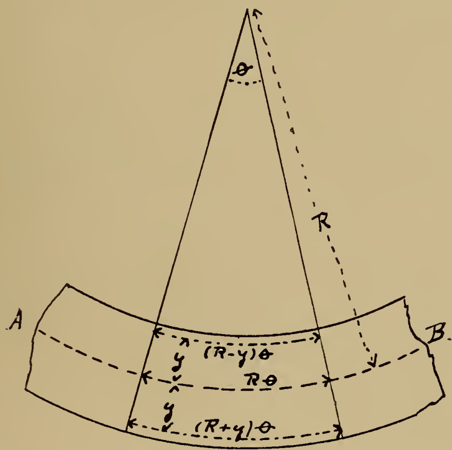

FIG. 16.

remain plane) will be inclined to each other, being nearer together on the concave side, the same distance apart as before bending at the neutral surface, and farther apart at the convex side. If R is the radius of curvature of the neutral line AB (Fig. 16), θ, the angle between the planes of any two sections, and y the distance from the neutral plane to any

point in the rod, then the length of the elastic curve between these two sections is $R\theta$, and the length between these two sections, along a line all points of which are at a distance y from the neutral surface is $(R \pm y)\theta$, or $R\theta \pm y\theta$, so that $y\theta$ is the *total* change of length between the two sections at a distance y from the neutral surface. The change of length per *unit* length is the total change divided by the original length, or the *strain* is $\frac{y\theta}{R\theta} = \frac{y}{R}$. From Chapter I, Art. 5, we have the strain due to tension or compression equal to $\frac{p}{E}$. Hence $\frac{p}{E} = \frac{y}{R}$, or the stress due to bending at a distance y from the neutral plane will be

$$p = \frac{E}{R}y.$$

The stress varies then as the distance from the neutral surface, and must not *at the outer surface* in the plane of bending exceed the elastic limit of the material, or the surface *fiber stress* must be less than f.

23. We have above our formula for the fiber stress, but do not know what value to use for y, as where the neutral axis lies is not known. Neither do we know the length of R. We will first prove that *the neutral axis of any section of a beam passes through the center of gravity of the section.* We know that the stress is maximum compressive at the surface on the concave side; decreases with the distance from the neutral surface, where it is zero; there changes to tension; and increases to a maximum at the surface on the convex side. The rod is in equilibrium, a law of which is that the sum of the horizontal components of *all* the forces acting must equal zero. Now the loads which cause the bending are all vertical, so can have *no* horizontal components, the only other forces are the horizontal stresses at the section, their sum must then

be equal to zero; the stresses on the opposite sides being equal but opposite in direction. Fig. 17 shows the stress on one side only, and the section may be of any shape whatever. Let AB be the neutral axis. The formula $p = \frac{E}{R} y$ gives us the stress per unit area on any element of area, dA, which is throughout at a distance y from the neutral axis. The stress on the element is then pdA, or $\frac{E}{R} ydA$. If we integrate this

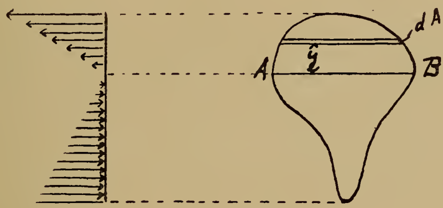

FIG. 17.

between the limits for our section, we will get the total stress on the section, which we know to be equal to zero, or

$$H = \frac{E}{R} \int_{\text{limit}}^{\text{limit}} ydA = 0. \qquad (1)$$

Now the value of *the integral*, $\int ydA$, divided by the area of the section will give us the distance of the center of gravity of the section from the line AB, and if this distance were zero the line would pass through the center of gravity of the section. Now equation (1) equals 0; we know E and R have finite values, therefore the integral $\int_{\text{limit}}^{\text{limit}} ydA$ must be equal to zero. Hence the neutral axis always passes through the center of gravity of the section.

24. Determination of R.—Our rod is in equilibrium, one of the laws of which is that the sum of the moments of *all* the forces about *any* axis must equal zero. Being true for *any* axis we will take the neutral axis for the axis of moments. We will first find the sum of the moments of the external forces. In Fig. 18 let AB be the rod of length l, loaded with W pounds in the middle, so that the supporting forces are each $\frac{W}{2}$; then to the left of the section HH, the moment of

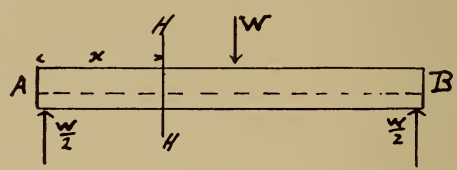

FIG. 18.

the supporting force $\frac{W}{2}$ is $\frac{W}{2}x$; and the moment of the forces to the right is

$$\frac{W}{2}(l-x) - W\left(\frac{l}{2} - x\right) = \frac{Wx}{2},$$

the same value as before. The moment to the left tends to turn the part of the beam on the left of the section in the direction of motion of the hands of a watch, while that to the right tends to turn the right part of the beam in the opposite direction; therefore, if we call the first direction positive, the other is negative. As in the other cases of equilibrium we will use the values found on one side only of the section. In this case, the moment of the external forces about the neutral axis is the *bending moment* for that section, and we will designate it by M, so $M = \frac{Wx}{2}$. Considering then the part to the left of this section, the moment of the stress in the sec-

tion must balance the moment of the external forces about the neutral axis. Fig. 19 shows enlarged the part of the rod to the left of the section. We see that the moment of the external forces tends to turn this part of the rod in the direction indicated by the arrow marked (1), while the moment of the stress in the section tends to turn it in the direction of the arrow (2), and for equilibrium these moments must be equal. As in Art. 23, the stress on the area dA is $\frac{E}{R}ydA$, and the

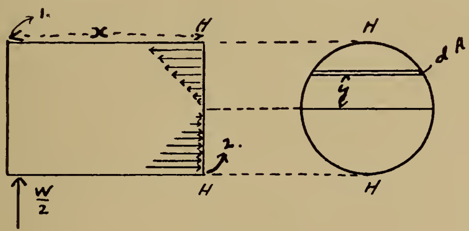

FIG. 19.

moment is $\frac{E}{R}ydA \, . \, y$, which integrated between the limits for the sections gives $\frac{E}{R}\int_{\text{limit}}^{\text{limit}} y^2dA$. The integral $\int y^2dA$ between limits is the moment of inertia of the area of the section about the neutral axis, and we will designate it by I, so that the moment of the stress in the section is equal to $\frac{E}{R}I$. This must equal M, so $M = \frac{E}{R}I$, or $R = \frac{E}{M}I$. In Art. 22 we found $\frac{E}{R} = \frac{p}{y}$, so we now have a general formula for bending,

$$\frac{p}{y} = \frac{M}{I} = \frac{E}{R}.$$

In which $p =$ stress at any point at a distance y from the neutral surface;
$M =$ bending moment at any section due to external loads;
$I =$ moment of inertia of the area of the section about the neutral axis;
$R =$ radius of the arc into which the rod is bent;
$E =$ modulus of elasticity of the material of the rod.

25. We have now seen the effect of applying forces to our rod in all the different ways. If the stresses caused have the *same line of action,* that is, if they act on any section in the same or in a diametrically opposite direction, their algebraic sum will give us the total stress on the section. For example, the stress due to several longitudinal loads is the algebraic sum of the loads divided by the area of the section; or, if a rod is under a tensile stress and there is also a fiber stress due to bending, the maximum stress would be at the convex surface and equal to the sum of the two stresses, while at the concave surface, which would be in compression due to the bending, the stress would be the difference between the two, and would act in the direction of the greater. Another fact must be considered in actual practice: When the temperature changes, a free rod expands or contracts without stress, but if the rod be *prevented* from expanding or contracting, *stress is produced.* The method of taking the algebraic sum when the lines of action are the same is called the *principle of superposition,* which may be stated as follows: The effect due to a combination of forces is equal to the sum of the effects due to each force taken separately.

When the stresses caused by our forces *have not* the same line of action, such as combinations of shearing or torsion with bending, we must arrive at their maximum effects in

some other way. In the next chapter we will endeavor to show how to find the maximum stress together with its direction, which is due to the combined effects of two or more stresses which act at right angles. We will then be able to combine the effects of any of the forces which we have applied separately to our rod in these first four chapters. We will find that the maximum stresses due to any of these combinations are greater than any of the stresses acting singly, and that they act on planes which are inclined to those on which any of the single stresses act. This will account for the apparently erratic manner in which material sometimes carries away; so in deciding if a single part of a machine or structure is strong enough, we must first find to what forces it is subjected, and then if it be able to sustain the total stresses they induce.

Examples:

1. A beam 2 ins. wide by 3 ins. deep is subjected to a bending moment of 72 ton-ins. What is the maximum fiber stress?

Solution:

$$\frac{p}{y}=\frac{M}{I}. \quad p=\frac{My}{I}. \quad y=\frac{3}{2}.$$

$$I=\frac{1}{12}Ah^2=\frac{9}{2}. \quad \therefore p=\frac{72\times 3\times 2}{2\times 9}=24 \text{ tons.}$$

2. An iron I-beam (without weight) of 12-ft. span has flanges 4 ins. by 1 in., and web 8 ins. by $\frac{1}{2}$ in. What is the greatest central load it can carry if the stress is limited to 4 tons per sq. in.?

Solution:

$$\frac{p}{y}=\frac{M}{I}. \quad M_{max}=\frac{Wl}{2}=W\times 3\,\text{ft.-lbs.}=W\times 36\,\text{in.-lbs.},\ I=\frac{542}{3}.$$

$$y=5. \quad \frac{4}{5}=\frac{W\times 36\times 3}{542}. \quad W=\frac{4\times 542}{5\times 36\times 3}=4.015 \text{ tons.}$$

3. A cast-iron I-beam has a top flange 3 ins. by 1 in.; bottom flange 8 ins. by 2 ins.; web, trapesoidal, $\frac{1}{2}$ in. thick at top and 1 in. thick at bottom; total depth of beam 16 ins. Find the position of the neutral axis and the ratio of maximum tensile to compressive stresses.

Ans. Neutral axis 4.81 ins. from bottom. Ratio T to C is 3 to 7.

4. A wooden beam of rectangular cross-section is 15 ft. long and 10 ins. wide. If the maximum bending moment is 16.5 ton-ft., and the allowed stress is $\frac{1}{2}$ ton per sq. in., what is its depth?

Ans. 15.4 ins.

5. An I-beam is 25 ft. long, top flange 3 ins. by 2 ins.; bottom flange 10 ins. by 3 ins.; web 12 ins. by 1 in.; total depth 17 ins. If the stress is limited to $4\frac{1}{2}$ tons per sq. in., find the greatest central load it can support in addition to its own weight (take weight of beam as 2000 lbs. acting at its center).

Ans. 6.48 tons.

6. What is the radius of the smallest circle into which a rod of iron 2 ins. in diameter may be bent without injury, the stress being limited to 4 tons per sq. in. $E = 13{,}000$ ton-ins.

Ans. $R = 270$ ft. 10 ins.

7. A spar, 20 ft. long, is supported at the ends and sustains a maximum bending moment of 3147.5 lb.-ft. If the stress be limited to $\frac{1}{2}$ ton per sq. in., what is the diameter of the spar?

Ans. 7.0025 ins.

8. What is the diameter of the smallest circle into which a $\frac{1}{2}$-in. steel wire may be coiled, keeping the stress within 6 tons per sq. in. (E for steel wire being 35,840,000 in in.-lb. units).

Ans. $111\frac{1}{9}$ ft.

9. A rectangular beam 12 ft. long, 3 ins. wide, 9 ins. deep, is supported at the ends. Stress is limited to 3 tons per sq. in. Find the load which can be carried at the center; also find the load if the beam lies the flat way, *i. e.*, 3 ins. deep and 9 ins. wide.

Ans. 1st case $3\frac{3}{8}$ tons; 2d case $1\frac{1}{8}$ tons.

10. Find the breadth and depth of the rectangular beam of maximum strength which can be sawed from a log 2 ft. in diameter, and compare its resistance to bending with that of the largest square beam that can be sawed from the same log.

Ans. Top is $\frac{2}{\sqrt{3}}$ ft., dept is $\frac{2\sqrt{2}}{\sqrt{3}}$ ft. Resist bending in ratio of 1.089 to 1.

11. A steel I-beam 30 ft. long; flanges 7 ins. by 8 ins.; web 24 ins. by .5 in.; carries 31,900 lbs. at its center. What is the maximum fiber stress?

Ans. 16,000 lbs. per sq. in.

12. Compare the resistance to bending of a wrought-iron I-beam; flanges 6 ins. by 1 in.; web 8 ins. by $\frac{3}{4}$ in., when upright, and when laid on its side.

Ans. 4.6 to 1.

13. A round steel rod, 2 ins. in diameter, can only withstand a bending moment of 6 ton-ins. What is the greatest length of such a rod which will just carry its own weight when supported at the ends?

Ans. $28\frac{1}{3}$ ft.

CHAPTER V.

COMBINATION OF STRESSES.

26. In a rod suffering tension and shear let HH (Fig. 20) be any section which has normal stress due to the tensile load, F, and tangential stress due to shear. Let the square shown represent the base of an elementary cube of volume, whose height dz, is perpendicular to the plane of the paper, and on

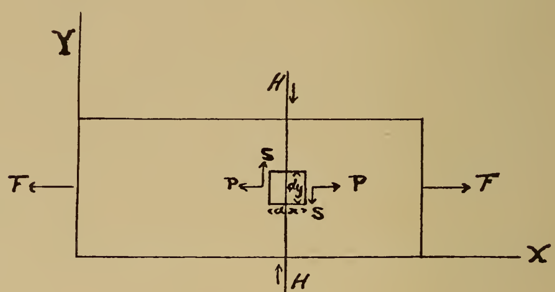

FIG. 20.

whose faces dy, dz, are the stresses P and S as shown. It is clear that the stresses P balance each other, but it will be noticed that the tangential stresses S form a moment whose effort is to turn the cube to the right. Now the forces are assumed within the elastic limit and we know the cube to be in equilibrium, so there *must* be an equal and opposite moment tending to turn it to the left. In other words, we must have tangential stresses equal to S on the faces $dxdz$, acting in a direction such that their moment will turn the cube to the right. These latter stresses are called *longitudinal shear*, and

because of the resistance offered to it a solid beam will bend less, when supported at the ends, than a pile of thin boards of the same volume. We may then concede that the *shearing stresses at any point within a body in a state of stress, are equal and act in planes at right angles with each other.*

27. We can now find the plane on which the resultant stress, due to a number of stresses acting in planes at right angles, is a maximum. Fig. 21 is, enlarged, the elementary

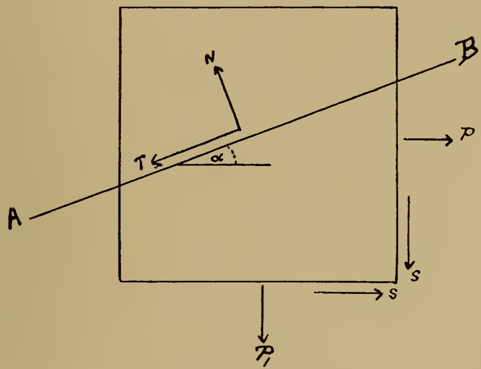

FIG. 21.

cube of Fig. 20, with the additional stress P_1, acting so as to put it in vertical tension. Let AB be any plane making the angle α with the direction of the stress P, and let the sum of the components of all the stresses on one side of this plane, when resolved along and perpendicular to it, be N and T as shown. The stresses on the other side of AB would give equal and opposite components to N and T (equilibrium). So that our cube is now in equilibrium under the stresses shown. Let us resolve these stresses *horizontally* and *vertically,* then the sum of both the horizontal and the vertical components will

be zero. The intensity of the stress P on AB is $P \sin \alpha$, etc. (see Art. 14). Resolving

$N \sin \alpha + T \cos \alpha - P \sin \alpha - S \cos \alpha = 0$ (1) horizontally.

$N \cos \alpha - T \sin \alpha - P_1 \cos \alpha - S \sin \alpha = 0$ (2) vertically.

Eliminating T between (1) and (2) we get

$$N = P \sin^2 \alpha + P_1 \cos^2 \alpha + 2S \sin \alpha \cos \alpha. \tag{3}$$

Reducing

$$\left(\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}\ ;\ \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}\right.$$

$$\left.\text{and } 2 \sin \alpha \cos \alpha = \sin 2\alpha\right)$$

$$N = \frac{P + P_1}{2} + \frac{P_1 - P}{2} \cos 2\alpha + S \sin 2\alpha. \tag{4}$$

By the same method we get

$$T = \frac{P - P_1}{2} \sin 2\alpha + S \cos 2\alpha. \tag{5}$$

(4) and (5) give us the stresses along and perpendicular to *any* plane due to the combined stresses P, P_1 and S. Putting equal to zero the first derivative with regard to N and α or T and α in (4) and (5), will give us the value of α for which N or T is a maximum or minimum.

$$\frac{dN}{d\alpha} = 0 = -\frac{P_1 - P}{2} .2 \sin 2\alpha + 2S \cos 2\alpha$$

gives

$$\tan 2\alpha = \frac{2S}{P_1 - P} \text{ or } \alpha = \tfrac{1}{2} \tan^{-1} \frac{2S}{P_1 - P} \left[\begin{matrix}+ \\ - \end{matrix} \frac{\pi}{2}\right. . \tag{6}$$

$$\frac{dT}{d\alpha} = 0 = \frac{P - P_1}{2} 2 \cos 2\alpha - 2S \sin 2\alpha$$

gives

$$\tan 2\alpha = \frac{P - P_1}{2S} \text{ or } \alpha = \tfrac{1}{2} \tan^{-1} \frac{P - P_1}{2S} \left[\begin{matrix}+ \\ - \end{matrix} \frac{\pi}{2}\right. . \tag{7}$$

28. Equation (6) gives us the angle with P of a plane on which the *normal* stress due to the combined load is a maximum (the plane 90° from it giving the minimum), and equation (7) the angle with P of a plane on which the *tangential* stress due to the combined load is a maximum (minimum 90° from it). Obviously, from these equations, the two planes are at 45° from each other ($\tan 2\alpha$ being in one case the negative reciprocal of what it is in the other), *or the planes of maximum normal* and *maximum tangential stresses lie at angles of* 45° with each other. Equation (6) shows the maximum and minimum normal stresses to be at right angles, and if we substitute the value of 2α found from it, in equation (5), we find that on this plane of maximum *normal* stress the tangential stress is zero, or *the shear is zero when the normal stress is a maximum.* The *normal* stresses found in equation (6) and acting on planes at right angles are called *principal stresses* and their directions *principal directions.* Principal stresses are always at right angles. If we substitute in equations (4) and (5), the values of $\sin 2\alpha$ and $\cos 2\alpha$, obtained from equations (6) and (7) respectively, we will get the maximum value of the normal and tangential stresses.

$$\left.\begin{array}{ll} N = \dfrac{P + P_1}{2} \pm \sqrt{4S^2 + (P - P_1)^2} & (8) \\ \text{and } T = \pm \frac{1}{2} \sqrt{4S^2 + (P - P_1)^2} & (9) \end{array}\right\} \begin{array}{l} + \text{ sign max.} \\ - \text{ sign min.} \end{array}$$

The $+$ sign of (8) gives the maximum *tension* (or compression), and the $-$ sign gives the maximum *compression* (or tension) on a plane at right angles, *i. e.,* principal stresses. Using the formula of this article we can find the maximum value and its direction of any combination of stresses due to the loads used in the preceding chapters.

29. The Stress Ellipse.—Let us assume that at any point within a body there are two normal stresses of intensity, P and P_1 (Fig. 22) acting at right angles. Then by Art. 14, x, the intensity of the stress P on any plane making an angle α with the direction of P is $P \sin \alpha$, and y, the intensity of the stress P_1 on the *same* plane, is $P_1 \cos \alpha$,

$$\frac{x}{P} = \sin \alpha, \text{ and } \frac{y}{P_1} = \cos \alpha,$$

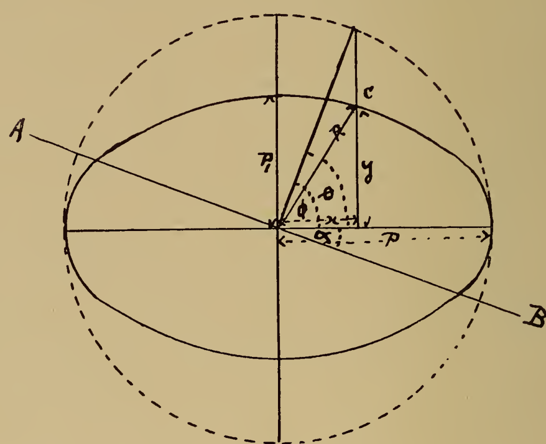

FIG. 22.

squaring and adding we have

$$\frac{x^2}{P^2} + \frac{y^2}{P_1^{\ 2}} = \sin^2 \alpha + \cos^2 \alpha = 1.$$

The above is the equation of an ellipse of which the semi-axes are P and P_1, and of which x and y, the coordinates of any point (C, Fig. 22), represent respectively the intensity of the stresses P and P_1 on a plane making the angle α with P. The radius vector, OC, represents on the same scale, the amount and direction of the *resultant intensity* of stress on the plane AB. The equation of this ellipse in terms of the

eccentric angle ϕ, is $x = P \cos \phi$, and $y = P_1 \sin \phi$. Now the tangent of the angle θ, which the resultant stress makes with the direction of P is

$$\tan \theta = \frac{y}{x} = \frac{P_1 \sin \phi}{P \cos \phi} = \frac{P_1 \cos a}{P \sin a} = \frac{P_1}{P} \tan \phi = \frac{P_1}{P} \cot a.$$

R, the resultant stress, is equal to $\sqrt{x^2 + y^2}$, and the angle the resultant stress makes with the *plane* AB is $(\theta + \alpha)$. Had there been another normal stress at right angles to the plane of P and P_1, the locus of the end of the resultant, C, would have been the surface of an ellipsoid. In this case it is probably as easy to find the resultant of two of the stresses and then get the resultant of the third with it. This ellipse is convenient to find the direction and amount of the resultant of the *principal stresses,* or the combinations of other nor-

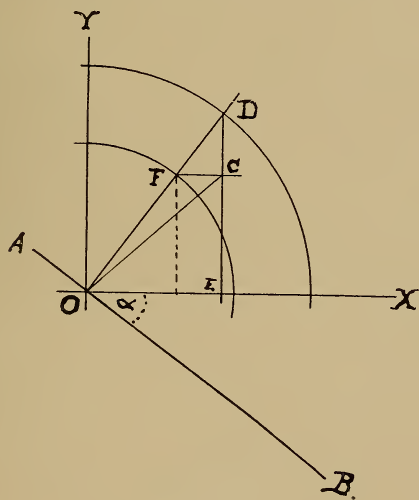

FIG. 23.

mal stresses, which may be done graphically as follows: Draw the axes OX and OY (Fig. 23) and lay off to the same

scale P on OX, and P_1 on OY. With these distances for radii, describe two concentric circles with centers at O. Draw the line AB, making the given angle α with P and draw OD perpendicular to it. Where OD cuts into the circle of radius P, drop a perpendicular on OX, and where it cuts the circle of radius P_1, draw FC parallel to OX. C, the intersection of these last two lines is a point on the ellipse. CE represents the stress due to P resolved on the plane AB, and OE represents the stress due to P_1, resolved on the plane AB; while OC represents the resultant stress on AB due to both, and the angle COB is the angle it makes with the plane AB.

30. Now any condition of stress at any point within a body may always be reduced (Art. 14) to three simple stresses acting on planes at right angles. By means of the formula of Arts. 27 and 28, we can always calculate the value and direction of the resultant stress on any plane due to these three simple stresses, and by means of the stress ellipse of Art. 29, we can calculate or determine graphically the amount and direction of the resultant of *two* simple normal stresses on any *given* plane. For example, if we find a piece of material under tensile stress combined with torsion, we can substitute for P in equation (8) the tensile stress, and for S the stress *at the surface* (maximum) due to torsion; and, unless there is another stress at right angles to the plane of these two, P_1 is zero. The value of N given by these substitutions is the maximum normal stress, and the plane on which it acts will be found by making the same substitutions in equation (6), remembering that α is the angle the plane makes with the direction of the tensile stress P. In case of bending, the *maximum* fiber stress should be used for P. We must always use the *maximum* values, for the piece of material must be strong enough to sustain the *greatest* stresses to which it will be subjected.

Example 5, at the end of this chapter, offers a good illustration of the use of the stress ellipse.

Examples:

1. The shaft of a vessel, 15 ins. in diameter, is subject to a twisting moment of 100 ft.-tons, and a bending moment of 20 ft.-tons, also the thrust of the screw is 16 tons. Find the maximum stresses on the shaft.

Solution:

$$P_1 = 0. \quad \therefore N = \frac{P}{2} + \tfrac{1}{2}\sqrt{4S^2 + P^2},$$

or

$$N(N - P) = S^2 \text{ and } T^2 = S^2 + \frac{P^2}{4}.$$

$$A = \frac{\pi d^2}{4}. \quad \therefore \text{thrust} = \frac{16}{\frac{\pi d^2}{4}} = \frac{64}{\pi d^2}.$$

$$p = \frac{My}{I} = \frac{64}{\pi d^2} \times 8. \quad \therefore P = \frac{64}{\pi d^2}(1 + 8).$$

$$q \text{ (torsion)} = S = \frac{Fa \text{ (twisting moment)}}{\frac{\pi d^2}{16}} = \frac{64 \times 20}{\pi d^2}.$$

$$N\left(N - \frac{64\,(1+8)}{\pi d^2}\right) = \left(\frac{64}{\pi d^2} \times 20\right)^2. \quad \therefore N = \begin{cases} 5.93 + \text{tons.} \\ 2.2626 + \text{tons.} \end{cases}$$

$$T^2 = \left(\frac{64 \times 20}{\pi d^2}\right)^2 + \left(\frac{64 \times 9}{2\pi d^2}\right)^2. \quad \therefore T = 1.8553 + \text{tons.}$$

2. A tube, 12 ins. mean diameter and $\frac{1}{2}$ in. thick is acted on by a thrust of 20 tons and a twisting moment of 25 ft.-tons. What are the maximum stresses? and their angles?

Ans. Greater 3.24 tons. $39\frac{1}{2}°$.

3. A rivet is under shearing stress of 4 tons per sq. in. and tensile stress, due to contraction, of 3 tons per sq. in. What are the maximum stresses?

Ans. Greater 5.8 tons.

4. The thrust of a screw is 20 tons; the shaft diameter 14 ins., has a twisting moment of 100 ton-ft., and a bending moment of 25 ton-ft. Find the maximum stress and compare it with what it would have been without the bending moment or thrust.

Ans. Greater 2.9 tons. Ratio 1.32 to 1.

5. At a point within a solid in a state of stress the principal stresses are tension of 255 and 171 lbs. Find the amount and direction of the stress on a plane making an angle of 27° with the 255 lbs. stress.

Ans. 191.35 lbs: 79° 46′ 20″.

6. A beam is under a 300-lb. tensile stress and a 100-lb. shearing stress. Find the normal and tangential stresses on a plane making an angle of 30° with the tensile stress.

Ans. Normal 161.5; tangential 179.8 lbs. per sq. in.

7. Find the principal stresses and their directions for a point in a beam which is under tensile stress of 400 lbs. and shearing stress of 250 lbs.

Ans. Normal maximum 520; minimum —120; $X =$ —25° 40′ 12″; or 64° 19′ 48″.

8. Find the maximum and minimum values of the shear in example 7 and their directions.

Ans. 320 lbs. per sq. in. $X = 19° \; 19' \; 48''$; 109° 19′ 48″.

9. A bolt, 1 in. in diameter, is under a tension of 5000 lbs. and a shearing force of 3000 lbs. Find the maximum stresses and their directions.

Ans. $N = 8155$; $T = 4970$ lbs. per sq. in. Angles are N 64° 53′, and T 19° 53′ with axis of bolt.

10. Find the maximum stress in a shaft 3 in. in diameter and 12 ft. between bearings, which transmits 40 H. P. at 120 revolutions per minute, and has a weight of 800 lbs. half way between bearings. The shearing stress due to above arrangement is 4000 lbs. per sq. in.

Ans. $N = 7600$ lbs. and $T = 4900$ lbs. per sq. in.

11. What is the diameter of a steel shaft to transmit 90 H. P. at 250 revolutions per minute, the distance between bearings to be 8 ft. and a load of 480 lbs. to be carried half way between bearings. The allowable maximum stresses being 7000 lbs. for N and 5000 lbs. for T (shearing force due to middle load 240).

Ans. $d = 2.8 +$ in. (about 3 ins.).

12. A steel bar of rectangular section, 18 ft. long, 1 in. thick, and 8 ins. deep, is under tension of 80,000 lbs., and a bending moment of 13,230 lb.-ins. What is the maximum stress in the bar?

Ans. 10,943 lbs. per sq. in.

13. A wooden beam 8 ft. long, 9 ins. deep, and 10 ins. wide, is under compression of 40,000 lbs., and a bending moment of 48,000 lb.-ins. What is the maximum stress?

Ans. 815 lbs. per sq. in.

CHAPTER VI.

SHEARING STRESS IN BEAMS.

31. In the preceding chapters we have made use of the simplest kinds of loads. In tension and compression we can vary the intensity of the stress only by increasing or diminishing the load as it has already been applied; in torsion also we can only change the value of the twisting moment by varying the amounts of the forces forming it or by changing the length of the arm; but to get the stresses due to bending and shearing we can apply loads in a number of different ways. We have also been neglecting the action of the force of gravity on our rod, which is of importance, as in the case of large shafts or beams whose bearings or supports must be spaced with reference to their weights; beside the supporting forces for a beam loaded in any way must be known in order to find the shearing and bending stresses.

32. Obviously, as we have assumed in Chapter IV, the supporting forces for a weightless, horizontal beam, loaded with a weight, W, exactly at its middle, are each $\frac{W}{2}$, or half the load; if, however, we take into consideration the weight of the beam and put several loads upon it at different places, the conditions of equilibrium require that the sum of the moments of *all the forces acting* on the beam, about *any* axis must be zero. *All the forces acting* are the loads, the weight of the beam *and* the supporting forces, all acting in the same plane.* If we take moments about an axis through one of

* If the forces do not all act in the same plane those acting in one plane may be considered separately, and the resultant of the

the supports, the moment of that supporting force will be zero, and the other supporting force, being now the only unknown quantity, is readily found by equating to zero the algebraic sum of the moments of all the forces; for example, it is required to find the supporting forces of a beam 20 ft. long, weighing 10 lbs. per ft., and supported at the ends, carrying 150 lbs. 6 ft. from the left end; 300 lbs. 11 ft. from the left end; and 750 lbs. 16 ft. from the left end (see Fig. 24). The weight of the beam is 200 lbs. and *acts at its center of gravity,* which in this case is at the middle of the beam. If we take moments about an axis through the left

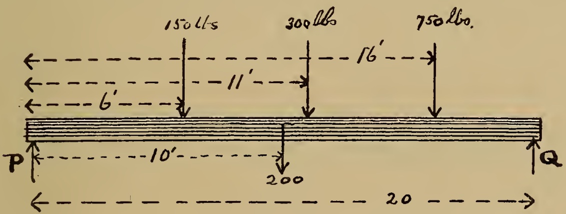

FIG. 24.

end of the beam, we eliminate the supporting force P, whose moment about this axis is zero; our equation for equilibrium will then be

$$200\times10+150\times6+300\times11+750\times16-Q\times20=0,$$

from which $Q=910$, and P is obviously the difference between the sum of the weights and Q, or $P=490$ lbs. In the above the weights act at the *points* indicated and are known as *concentrated* loads. If the loading is continuous the effect is considered as acting at the center of gravity, as it has been taken for the weight of the beam in the above.

---

several supporting forces in the different planes found. The supports, however, must be strong enough in the directions of each of these planes to sustain the forces acting in those planes.

33. Having the supporting forces, we can now find the shearing stress on any transverse section of a beam. Referring to Fig. 25, and at first considering the beam weightless, if HH is any section between the supporting force P and the first load W_1, there would clearly be a tendency for the two parts of the beam to move as shown, the left part remaining stationary and the right part being pushed down by the weights W_1, W_2, and W_3; in other words there is in any loaded beam tangential or shearing stress along any transverse section. The magnitude of this stress (see Chap. II) is equal to the load *on one side* of the section, divided by the area of the section. In the beam of Fig. 25, the only load on

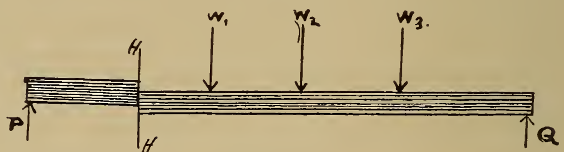

FIG. 25.

the left of the section HH, is the supporting force P, therefore, the shearing stress on any section between P and the first load W_1, is $\frac{P}{A}$.* Had we taken our section between W_1 and W_2 we would have had two loads on the left of the section and acting in opposite directions, so that the shearing stress would have been $\frac{P - W}{A}$, and so on across the beam, showing a drop as each load is passed. We will assume as positive the conditions as in Fig. 25, the tendency being

* In Chapter II we took our shearing forces indefinitely near together to get *pure* shear, that is, shear without bending. Shear and bending are closely connected, so that it is difficult to produce the one without the other.

for the part of the beam to the right of the section to move down; then let us construct a curve of shearing force using for ordinates the values as found above. For example, Fig. 26 is a beam loaded as shown. Taking the left end as origin, the total shearing force at that end is equal to the supporting force there, or S. F. $= P = 29$ lbs., which it remains until, as we move the section to the right, we get to the first

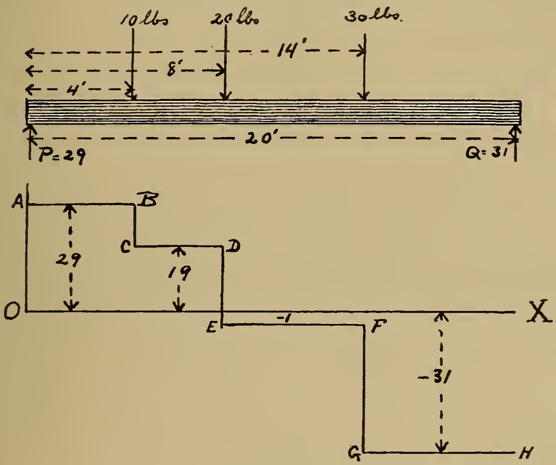

FIG. 26.

load, 10 lbs., which acts in the opposite direction to the supporting force; so, consequently, as we pass this load the total shearing stress drops to 19 lbs., where it remains until we come to the 20-lb. load. Here it drops to —1 lb. At this point the tendency for the right part of the beam to move down ceases, as the sum of the loads on the left side of a section from this point on will be greater than the supporting force at the left end. Passing the 30-lb. load, the shearing stress again drops, this time to —31 lbs. and the curve of shearing force will be the series of steps *ABCDEFGH*.

These values, with their signs, are for the left side of the section; on the right side the stresses are equal to those found but opposite in direction. Had we used the right end for origin and the left side down as the positive direction, our curve would have been a series of steps down to the left, with numerical values the same as before.

34. If we take the weight of the beam or, what is the same thing, consider the beam in the preceding article to be uniformly loaded all along its length, the value of the shearing

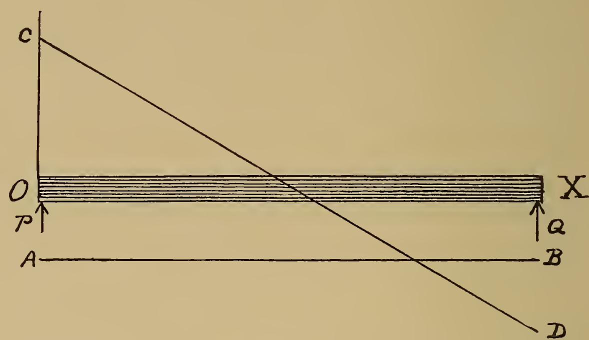

FIG. 27.

force will change at each point as we move our section to the right. Let Fig. 27 represent a beam so loaded; then if the ordinate of the line AB represents the load on unit length of beam, AB will be the load curve, being drawn below the beam or having negative ordinates because the load acts down. The supporting forces are now each half the total load, and at the left end the shearing force equals the supporting force P; but as we move the section to the right we reduce the shearing force by the amount of the load between the section and the supporting force, the S. F. being equal to $P - wx$, where w is the load per unit length, and x the distance from

the left end. The shearing curve will therefore drop steadily, as in the figure. The S. F. curve being CD. If we have in addition to the uniform load several concentrated loads, the curve of shear will be as in Fig. 28, with a drop at each concentrated load. If drawn to scale, these curves will give the shearing force at any point of a beam by measuring the ordinate at that point. As a general rule, then, the shearing

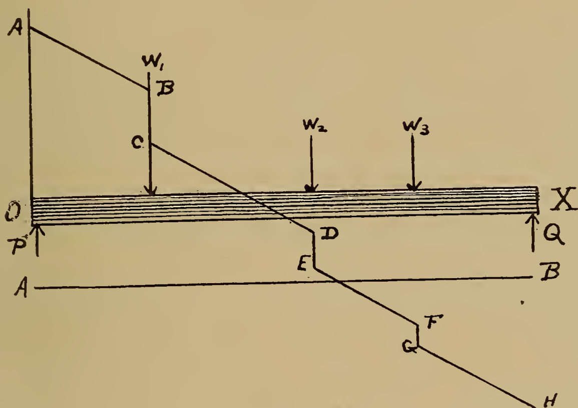

Fig. 28.

force at any transverse section of a loaded beam is equal to the algebraic sum of all the loads acting *on one side* of that section.

35. Hereafter we will use the left end of the beam for the origin in the equations for curves of any kind. With continuous loads, L, the load at any point will vary in some way with the distance from the origin; for example, a beam "a" ft. long carries a load which uniformly increases from zero at the origin to w lbs. per ft.-run at the right end. Here the load per ft.-run at a distance x from the origin would be

$\frac{wx}{a}$, found from the ratio $\frac{L}{x} = \frac{w}{a}$ and the equation of the load curve (w acting downward) would be $L = -\frac{wx}{a}$. For a uniform load of w lbs. per ft.-run, $L = -w$. The equations for load curves are easily found and are very useful, for let us notice the relation between the loads and the shearing force at any section. As a general formula for *concentrated loads,* we found the S. F. at any section to be equal to $P - \Sigma W$. With a beam loaded with any *continuous load,* if L be the load per unit length, then Ldx is the load on any elementary length, dx, of the beam and the total load from the origin to a section at any distance, x, from the origin is $\int_0^x Ldx$, the value of L being given by the equation to the load curve. Now the value of this integral is the algebraic sum of all the loads to the left of the section, and that is also our definition of shearing force. As a general rule, then, for any continuous load the shearing force is

$$\text{S. F.} = \int_0^x Ldx.$$

In using the formula we must remember that the *constant* of *integration* is the value at the origin of the quantity, represented by the integral, in this case that is the value of the supporting force P.

For *concentrated* loads we must still use S. F. $= P - \Sigma W$, and if a *continuously* loaded beam supports also a number of concentrated loads we must find the shearing forces separately, and get the total shearing force by algebraically adding the results (principle of superposition). In this latter case also the *supporting* forces for both the continuous and concentrated loads must be found and used separately.

Examples:

1. A beam 15 ft. long is supported at the ends and carries 4 tons 5 ft. from the left end, 1 ton 8 ft. from the left end, and $\frac{1}{2}$ ton 10 ft. from the left end. Find the supporting forces and draw the shearing curve.

Ans. $P = 3.3$ tons. $Q = 2.2$ tons.

2. A spar 20 ft. long is supported at the ends and loaded with 500 lbs., 4 ft.; 250 lbs., 9 ft.; and 900 lbs., 18 ft. from the left end. Find the supporting forces and draw the shearing curve.

Ans. $P = 627.5$ lbs. $Q = 1022.5$ lbs.

3. A plank 16 ft. long is laid across a ditch and a man weighing 192 lbs. walks across it. Find the shearing curve when he is 3 and when he is 8 ft. from the left end.

4. A weight of 384 lbs. is placed 5 ft. from the left end of the above plank. What is the shearing force at a point $6\frac{1}{2}$ ft. from the left end when the man is 8 ft. from there?

Ans. —24 lbs.

5. A beam 18 ft. long has a uniform load of 50 lbs. per ft. run. Draw curve of shearing force and give value at points 7 ft. and 13 ft. from the left end.

Ans. At 7 ft. 100 lbs.; at 13 ft. —200 lbs.

6. On the beam of example 5, weights of 300 lbs. and 500 lbs. are placed 6 ft. and 12 ft. respectively from the left end, in addition to its uniform load. Draw S. F. curve and give value of S. F. 8 ft. from left end.

Ans. $116\frac{2}{3}$ lbs.

7. A beam 5 ft. long has its left end fixed in a wall and supports a load of 1000 lbs. at its free end. What is the S. F.? and draw curve.

8. The beam of example 7 has a distributed load of 100 lbs. per ft.-run. What is the maximum S. F.? and draw the curve.

9. An oak beam 15 ft. long and 1 ft. square floats in sea water. It is loaded at the center with a weight which will just immerse it wholly. Draw curve of S. F., and give maximum value. 35 cu. ft. of sea water weighs 1 ton; 1 cu. ft. of oak weighs 48 lbs.

Ans. Maximum S. F. = 120 lbs.

10. A pine beam 20 ft. long and 1 ft. square floats in sea water, and is loaded at the middle with a weight which will just immerse it. Draw a curve of S. F. and give value 5 ft. from left end.

Ans. S. F. 5 ft. from left end = 125 lbs.

11. A beam 20 ft. long, supported at the ends, carries a load which uniformly increases from 0 at the left end to 50 lbs. per ft.-run at the right. Draw the curve of S. F., and find its maximum value, also find the point of the beam where its value is zero.

Ans. Maximum value = $333\frac{1}{3}$ lbs. Value zero where $x = 11.5 +$ ft.

12. The buoyancy of an object floating in the water is 0 at the ends and increases uniformly to the center, while its weight is 0 at the center and increases uniformly to the end. Draw curve of S. F., and give maximum value.

Ans. Maximum S. F. $= \frac{W}{4}$.

CHAPTER VII.

CURVES OF BENDING MOMENTS AND SHEARING FORCE.

36. In Chapter II it has been shown that when a beam is loaded there are horizontal stresses set up on any transverse section, these stresses being compressive on one side and tensile on the other side of the neutral plane. Art. 14 of the same chapter shows that the moment about the neutral axis of the *external forces* on one side of any section must be equal

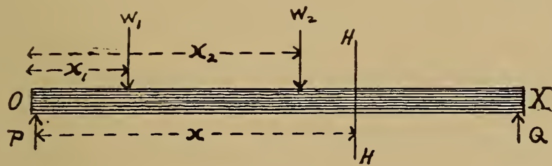

FIG. 29.

to the moment about the neutral axis of the stresses on the same side of the section. Definition: The *bending moment* at any section is the moment about the neutral axis of that section of all the *external* forces *on one side* of that section.

By definition then the bending moment at the section HH of a beam loaded as in Fig. 29 would be

$$M = Px - W_1(x - x_1) - W_2(x - x_2),$$

or finding the line of action of the resultant of all the forces W_1, W_2, etc., and calling its distance from the section "d"

$$M = Px - \Sigma W . d,$$

which is true for any kind of load.

If now we consider the increment ΔM which the bending moment receives if we take our section a distance Δx to the

right, the bending moment about the neutral axis of the new section will be

$$M + \Delta M = P(x + \Delta x) - \Sigma W(d + \Delta x)$$
$$= Px - \Sigma W \,.\, d + \Delta x(P - \Sigma W),$$

but $Px - \Sigma Wd$ is the original bending moment equal to M, and $(P - \Sigma W)$ is our formula for the shearing force (F) at any section, so

$$M + \Delta M = M + F\Delta x, \text{ or } \Delta M = F\Delta x,$$

and passing to the limit $dM = F \,.\, dx$; integrating, we have for a general formula for bending moment

$$M = \int F \,.\, dx.$$

This equation is true for any kind of load, but care must be taken to use the correct constant of integration which is the value of M at the origin. This for beams *free* at the origin is zero, but if a beam is *fixed* (prevented from moving in any way) at the origin, there *is* a bending moment there. We will devote the rest of this chapter to some examples of shearing force and bending moments in beams loaded and supported in different ways.

37. A beam supported at the ends, is loaded as shown in Fig. 30. Find the curves of shearing force and bending moment. By definition the S. F. $= P - \Sigma W$, by which we get the curve of S. F. to be $ABCDEF$ (Fig. 30, *a*). Considering each load separately, the supporting forces for the 25-lb. load are $P = 17.5$, $Q = 7.5$, and the S. F. curve is $ABCD$ (Fig. 30, *b*). The 35-lb. load gives supporting forces $P = 7$, $Q = 28$, and the S. F. curve is $ABCD$ (Fig. 30, *c*). If we add algebraically the ordinates given by these curves at any point distant x from the origin we will get the ordinates for the S. F. curve for that point given in Fig. 30, *a*.

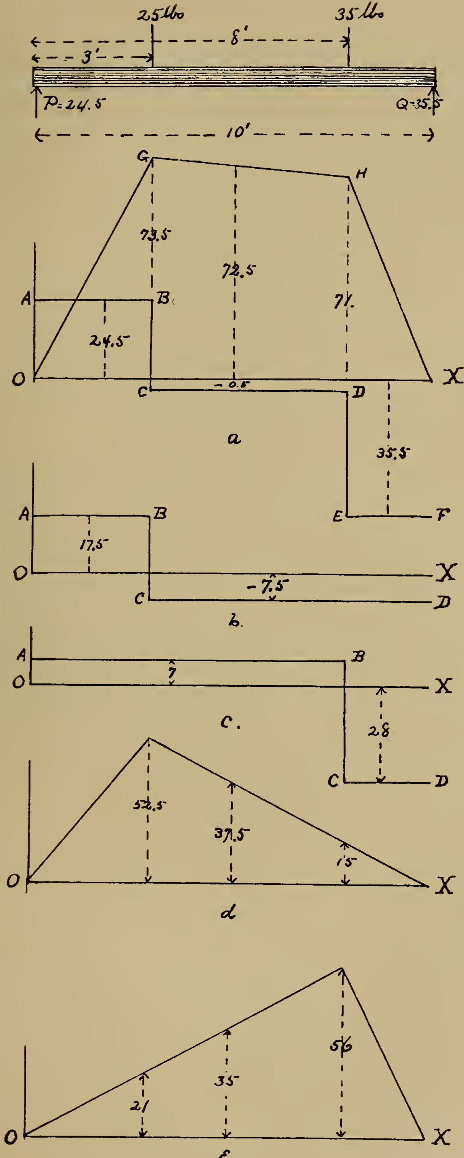
25 lbs
35 lbs
8'
3'
P=24.5
Q=35.5
10'
G
H
73.5
72.5
71.
A
B
24.5
O
X
C
-0.5
D
a
35.5
E
F
A
B
17.5
O
X
C
-7.5
D
b.
A
B
7
O
X
c.
28
C
D
52.5
37.5
15
O
X
d
56
35
21
O
X
e.

FIG. 30.

Obtaining the ordinates by definition, the curve of B. M. for the beam with these loads is given by *OGHX* in Fig. 30, *a*.

Let us get separately the curves of B. M. due to each load. For the 25-lb. load we get by definition the curve in Fig. 30, *d*, the maximum B. M. being directly under the load and the curve being positive at all points. For the 35-lb. load we get the curve in Fig. 30, *e*, this curve also being positive at all points. If we add together the ordinates given by these curves at any point distant x from the origin we will get the ordinate of the curve of B. M. for that point as given in Fig. 30, *a*. Now by definition, the bending moment at the middle of the beam, for example, is

$$24.5 \times 5 - 25 \times 2 = 72.5 \text{ lb.-ft.}$$

and by the formula $\int F \, . \, dx$ it is $-.5 \times 5 = -2.5$, which is evidently incorrect, but is due to the fact that having taken the two *concentrated* loads together, the shearing force of one being positive and the other negative, we have not multiplied the total shearing force by x. The moments of the two shearing forces cause bending in the *same direction*, therefore, if we neglect the negative sign (which we have *assumed* to indicate direction only) the shearing force at this point will be $7 + 7.5 = 14.5$ which multiplied by 5 gives 72.5 as before. With many loads to get the shearing force separately would be a tedious operation, so it is better to *make it a rule* to get the S. F. and B. M. for *concentrated* loads from the definition. We will find no trouble with beams having *continuous* loads unless there are concentrated loads in addition, in which case we get the curves due to the continuous loads by formula, and those due to the concentrated loads by definition, and apply the principle of superposition.

38. A beam 10 ft. long supported at the ends is loaded with a uniform load of 25 lbs. per ft.-run. Find the curves of

S. F. and B. M. In this case, a continuous load, our formula is very convenient. $L = - w = - 25$ lbs.

$F = \int L dx = - 25x + C$, C being the S. F. at the origin is equal to $P = 125$ lbs. $\therefore\ F = 125 - 25x$. (1)

(1) is the equation to the curve of S. F. This being an equation of the first degree is a straight line and plots as the line AB of Fig. 31. The bending moment is

$$M = \int F dx = \int (125 - 25x)\, dx,$$

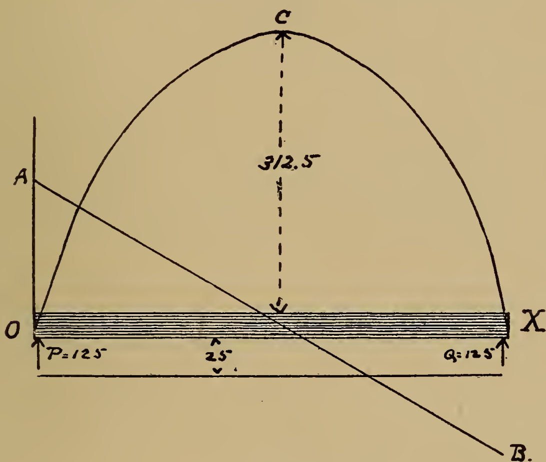

FIG. 31.

or,

$$M = 125x - \frac{25x^2}{2} + [C = 0, \qquad (2)$$

B. M. is zero at the origin.

(2) is the equation of the curve of B. M., and being of the second degree is a conic (parabola) and plots as OCX of Fig. 31. The maximum B. M. is at the middle of the beam where the S. F. is zero. To get values for either S. F. or B. M. at any section of the beam, substitute for x the distance of the section from the origin in equations (1) or (2) respectively.

39. A beam 5 ft. long, fixed at the left end, with the right end unsupported, carries a weight of 50 lbs. on the right end. Find the curves of S. F. and B. M.

In this case the left end supports the whole load, so $P = 50$, and the shearing force by definition (concentrated loading) is $F = 50$ lbs. The curve being the straight line A. B. of Fig. 32. The bending moment by definition is $M = Px$, which being an equation of the first degree is a straight line, the ordinates varying from zero to 200 lb.-ft. This would give us a line inclined upward from O, but we *know* the great-

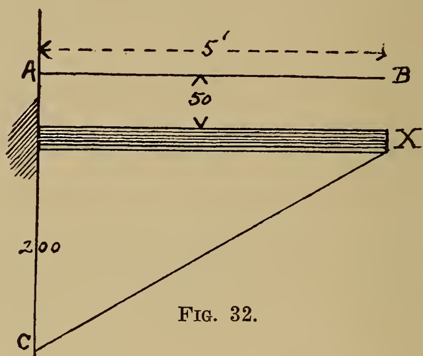

FIG. 32.

est B. M. is at the origin. Now notice that the curvature of this beam is just the opposite of that of a beam supported at the ends, the center of curvature being *below* the beam in this case, while it is *above* a loaded beam supported at the ends. In fact if we turn Fig. 32 "upside down" we will have just one-half of the beam supported at the ends and loaded with $2P$ in the middle, W being one of the supporting forces; so this kind of bending is called *negative*, and instead of the curve of B. M. inclining upward from O, it inclines downward from X as in the figure. The curve plots directly if we take the origin at X and move the section to the left, for this arrangement is the same as a beam twice as long supported in the middle and loaded with 50 lbs. at both ends.

40. A beam 5 ft. long fixed at the left end, with the right end unsupported carries a uniform load of 25 lbs. per ft. run. Find the curves of S. F. and B. M.

A continuous load; so we will use the formula

$$L = -w = -25 \text{ lbs.}$$

$$F = \int L dx = -25x + C.$$

C being the S. F. at the origin is equal to $P = 125$ lbs.

$$\therefore\ F = 125 - 25x. \tag{1}$$

(1) being an equation of the first degree is a straight line and plots as AX in Fig. 33. The bending moment is

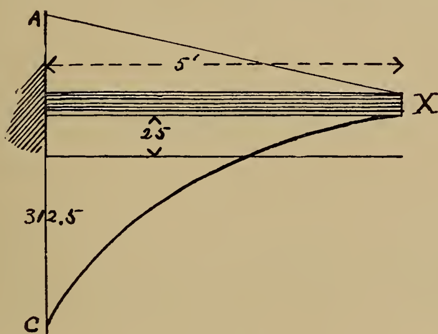

FIG. 33.

$$M = \int F dx = \int (125 - 25x)\, dx = 125x - \frac{25x^2}{2} + C_1.$$

Here C_1 is *not* zero, but we know there is no bending moment at the right end of the beam so we substitute $M = 0$ and $x = 5$, and solve for C_1

$$0 = 125 \times 5 - \frac{25 \times 25}{2} + C_1,$$

from which $C_1 = -312.5$, substituting this value of C_1 we get

$$M = 125x - \frac{25x^2}{2} - 312.5 \tag{2}$$

for the equation of the curve of B. M., which being of the second degree is a conic (parabola) and plots as XC in Fig. 33.

This equation gives a maximum negative value of B. M. at the origin, where we know it should be.

41. A beam 10 ft. long, supported at the ends, is loaded with a uniform load of 25 lbs. per ft.-run, and a concentrated

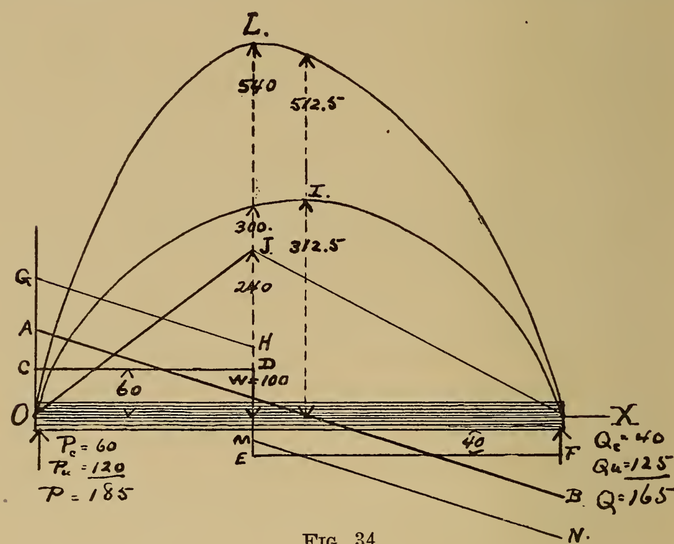

FIG. 34

load of 100 lbs. 4 ft. from the left end. Find the curves of S. F. and B. M.

For the uniform load (Art. 38) the curve of S. F. is AB (Fig. 34). For B. M. the curve is OIX.

For the concentrated load the S. F. curve is $CDEF$, and the curve of B. M. is OJX.

Adding algebraically the respective ordinates, we get the S. F. curve for both loads to be $GHMN$, and for B. M. the curve of both loads is OLX.

42. A beam 10 ft. long is supported at the ends and has a load uniformly increasing from zero at the left end to 100 lbs. per ft.-run at the right end. Find curves of S. F. and B. M. Here (see Art. 35)

$$L = -\frac{100x}{10} = -10x,$$

$$\text{S. F.} = \int L dx = \int -10x dx = -\frac{10x^2}{2} + C,$$

where C being the S. F. at the origin is equal to P. We can find P by the method used in the example at the end of Art. 32, as follows: In Fig. 35 OL is the load curve, the center

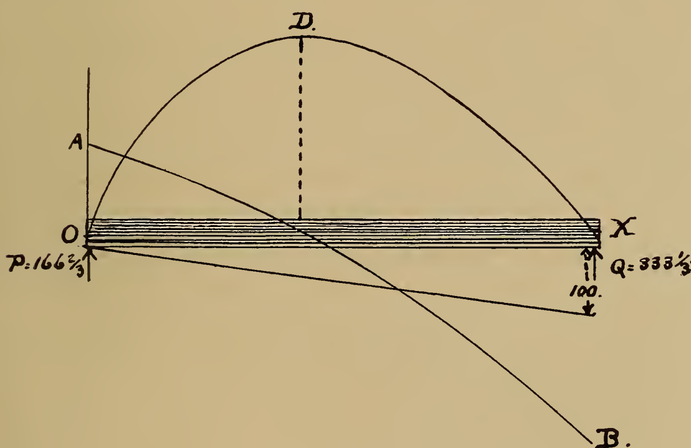

FIG. 35.

of gravity of the load then acts at $\frac{2}{3}$ the length of the beam from O, and the total load is $100 \times \frac{10}{2} = 500$ lbs., therefore, taking moments about O

$$500 \times \tfrac{2}{3} \cdot 10 - Q \times 10 = 0;$$

and $Q = 333\frac{1}{3}$, the other supporting force being $500 - 333\frac{1}{3}$; or $P = 166\frac{2}{3}$. But we can get P in another way, for

$$\text{S. F.} = P - 5x^2$$

and

$$\text{B. M.} = \int F dx = \int (P - 5x^2)\, dx = Px - \frac{5x^3}{3} + C_1,$$

C_1 is equal to zero, for the bending moment at the origin is zero, and M is also zero at the right end of the beam, so that if we substitute $x = 10$ and solve the equation

$$P \times 10 - \frac{5 \times 1000}{3} = 0$$

we get $P = 166\frac{2}{3}$ as before. Putting this value of P in the equation for S. F. and B. M. we get

$$\text{S. F.} = 166\tfrac{2}{3} - 5x^2 \tag{1}$$

and

$$\text{B. M.} = 166\tfrac{2}{3}x - \frac{5x^3}{3}, \tag{2}$$

which equations give the curves ACB for shearing (Fig. 35) and ODX for bending.

If we put (1) equal to zero and solve for x we get the point on the beam where the shearing force is zero; and if we put this value of x in (2) we will get the maximum B. M., for the maximum B. M. occurs where the S. F. is zero (see Art. 28), for by the principles of maxima and minima, the first derivatives of the B. M. with respect to x, put equal to zero will give us the shearing equation (1) equal to zero

$$\frac{d\,(BM)}{dx} = 166\tfrac{2}{3} - 5x^2 = 0;\ \text{from which } x = \sqrt{\frac{500}{3 \times 5}} = \frac{10}{\sqrt{3}}$$

the value of x where B. M. is a maximum.

Examples:

1. A beam 10 ft. long, supported at the ends, carries a load of 1000 lbs. 4 ft. from the left end. Find curves of S. F. and B. M.

2. A beam 5 ft. long, fixed at the left end and unsupported at the right end, carries 1000 lbs. at the right end. Find curves of S. F. and B. M.

3. A beam 10 ft. long, supported at the ends, carries a uniform load of 100 lbs. per ft.-run. Find curves of S. F. and B. M.; give values at ends and center.

Ans. S. F., ends $\pm$ 500; center 0. B. M., ends 0; center 1250 lb.-ft.

4. A beam 5 ft. long, fixed at the left end, unsupported at the right end, carries a uniform load of 100 lbs. per ft.-run. Find curves of S. F. and B. M.; and give values at ends.

Ans. S. F., left end 500 lbs.; right end 0. B. M., left end —1250 lb.-ft.; right end 0.

5. A beam 10 ft. long, supported at the ends, carries 400 lbs. 4 ft. from the left end, and 600 lbs. 6 ft. from the left end. Find curves of S. F. and B. M., and give values at the ends and center.

Ans. S. F., left end 480 lbs.; right end 520; center 80. B. M., left end 0 lbs.; right end 0; center 2000 lb.-ft.

6. What is the longest steel bar of cross-section of 1 sq. in. that can be supported at its center without being permanently bent, the greatest allowable bending moment for the bar being 2000 lb.-ins.?

Ans. $19\frac{11}{12}$ ft.

7. A beam 20 ft. long, supported at the ends, carries 2000 lbs. 5 ft. from the left end, and 5000 lbs. 4 ft. from the right end. Find curves of S. F. and B. M., and give values at the ends and at the loads.

Ans. S. F., left end 2500 lbs.; right end 4500 lbs.; be-

tween loads 500 lbs. B. M., end 0; first load 12,500 lb.-ft.; second load 18,000 lb.-ft.

8. If the beam of example 7 were loaded with 200 lbs. per ft.-run, find the curves of S. F. and B. M. and give values at ends and center.

Ans. S. F., ends $\pm$ 2000 lbs.; center 0. B. M., ends 0; center 10,000 lb.-ft.

9. A round steel rod of 2 ins. diameter can only withstand a bending moment of 6 ton-ins. What is the greatest length of such a rod which will just carry its own weight when supported at the ends?

Ans. 29.2 ft.

10. A beam 20 ft. long, supported at the ends, carries a uniformly distributed load of 5 tons, and a concentrated load of 5 tons, 4 ft. from the left end. Find curves of S. F. and B. M. and give values at the center. Where is the greatest bending moment and give its value?

Ans. S. F., center -1 ton. B. M., center $22\frac{1}{2}$ ton-ft. B. M., maximum $24\frac{1}{2}$ ton-ft., 6 ft. from left end.

11. A pine beam 20 ft. long and 1 ft. square floats in sea water. It is loaded at the center with a weight just sufficient to immerse it wholly. Find curves of S. F. and B. M., and give maximum values. A cu. ft. of pine weighs 39 lbs., of sea water 64 lbs.

Ans. Maximum S. F., 250 lbs. Maximum B. M., 1250 lb.-ft.

12. A beam 54 ft. long, supported at the ends, is loaded with 15 cwt. per ft.-run, for a distance of 36 ft. from the left end. Find the curves of S. F. and B. M. What is the maximum B. M. and the B. M. at 6, 12, and 36 ft. from the left end?

Ans. B. M. maximum $=$ 216 ton-ft.; B. $M._6 = 94.5$ ton-ft.; B. $M._{12} = 162$ ton-ft.; B. $M._{36} = 162$ ton-ft.

13. A steel beam 5 ft. long is fixed at one end, unsupported at the other end, and is of rectangular section 2 ins. wide and 3 ins. deep. What weight at the free end will destroy the beam if the limiting stress is 24 tons per sq. in.

Ans. 1.2 tons.

14. The buoyancy of a floating object is 0 at the ends, and increases uniformly to the center, while the weight is 0 at the center and increases uniformly to the ends. Find the curves of S. F. and B. M. and give maximum values in terms of the displacement, D, and the length, l, of the object.

Ans. S. F. maximum $= -\frac{D}{4}$; B. M. maximum $= -\frac{lD}{12}$.

CHAPTER VIII.

SLOPE AND DEFLECTION.

43. The *slope* at any section of a loaded beam is the angle between the tangent to the *neutral* line at that section and the straight line with which the *neutral line* would coincide if the beam were not bent; or, *slope* is the angle between our axis of X and the tangent at any section to the curve into which the beam is bent.

The *deflection* at any section of a loaded beam is the distance from the axis of X to the point where the *neutral line* pierces that section; or, it is the ordinate at that section of the *neutral line* (see Fig. 36).

From Chapter IV we have the general formula for bending

$$\frac{M}{I}=\frac{E}{R}, \text{ from which } R=\frac{EI}{M}.$$

E being a constant and I also, in this case, as we are considering beams of uniform cross-section; M, of course, varies at different sections of the beam.

By calculus the formula for the radius of curvature at any point of a curve given by its rectangular equation is

$$R=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Equating these two values of R we have

$$\frac{E.I.}{M}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}, \qquad (1)$$

which by integration will give us the equation to the curve into which the beam is bent. As this integration would be somewhat complicated, and as in properly built structures the dimensions of the different pieces of material are such that the bending is very slight, we can without appreciable error simplify the operation considerably as follows: $\frac{dy}{dx}$ being the tangent of the angle that the bent beam at any point makes with the axis of X, is a very small fraction, and being *squared* in equation (1) becomes so small that it may be neglected, in comparison with unity, so that equation (1) for all practical purposes becomes

$$\frac{EI}{M} = \frac{1}{\frac{d^2y}{dx^2}}; \text{ or, } EI\frac{d^2y}{dx^2} = M;$$

and integrating this equation we get

$$EI\frac{dy}{dx} = \int M dx, \tag{2}$$

which will give us the tangent of the slope at any point when we substitute for M its value in terms of x as found in Chapter VII, and integrate. The integral, $\int Mdx$ is called the slope function, and we will designate it by S. When S is divided by EI we have the tangent of the angle of slope.

If we write equation (2)

$$EI\frac{dy}{dx} = S,$$

and integrate again we get

$$EIy = \int S dx. \tag{3}$$

This is the equation of the curve into which the beam is bent and the value of y given by this equation is the ordinate of the neutral line at any section distant x from the origin. y is usually negative.

44. In integrating to get values for equations (2) and (3) we must not forget the *constants* of *integration.*

If a beam is fixed at the origin, the slope there is zero, but a beam supported at the ends and loaded will bend into a curve like that of Fig. 36. Clearly the slope is greatest at the ends and there will be a constant of integration for equation (2). We will usually know from the way in which the beam is loaded a value of x, at which the slope is zero.

For example, with a *uniform load* the beam of Fig. 36 would bend so that *at its middle* the slope would be zero, therefore, if we substitute (having integrated the right mem-

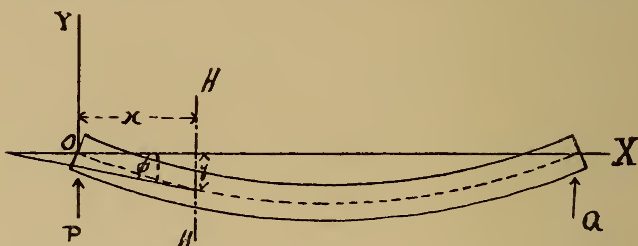

FIG. 36.

ber) for x, in equation (2), half the length of the beam and put the equation equal to zero, we can solve for C, the constant. If we do not know a value of x where the slope is zero, we will know a point where the *deflection* is zero (one of the supports for example), and substituting in *equation* (3) the value of x for this point, solve for the constant of integration of *equation* (2). An example will be solved illustrating the method.

There is usually no difficulty in finding the constant of integration for equation (3), because the *deflection* of a beam at the origin, whether the beam be fixed or supported there and no matter how loaded, is zero. Of course a beam could be propped up somewhere along its length, so that the left end would not rest on its support.

45. A beam, 10 ft. long, supported at the ends, is loaded with a uniform load of 25 lbs. per ft.-run. Find the slope and deflection.

Here the bending is perfectly symmetrical. Let Fig. 36 represent the bent beam, O being the origin and OX the position of the neutral line *before* the load was applied, the dotted line representing its position after the application of the load. $P = Q = 125$.

$$L = -25,$$

$$F = -25x + C = -25x + 125. \quad [C = P],$$

$$M = -\frac{25x^2}{2} + 125x + [C = 0],$$

$$EI\frac{dy}{dx} = -\frac{25x^3}{6} + \frac{125x^2}{2} + C_1. \tag{1}$$

Knowing the bending is symmetrical, $\frac{dy}{dx} = 0$ where $x = 5$ (the middle), so

$$0 = -\frac{25 \times 125}{6} + \frac{125 \times 25}{2} + C_1 \dots \quad \therefore C_1 = -1041\tfrac{2}{3}.$$

In case we do not know the bending is symmetrical, we carry C_1 down through the next integration for deflection which gives

$$EIy = -\frac{25x^4}{24} + \frac{125x^3}{6} + C_1 x + [C_2 = 0. \tag{2}$$

C_2 is zero for the deflection is zero at the origin. The deflection is also zero where $x = 10$ (the other support), and substituting this value we get

$$0 = -\frac{25 \times 10000}{24} + \frac{125 \times 1000}{6} + 10C_1,$$

from which $C_1 = -1041\frac{2}{3}$ as before, and equation (1) becomes

$$EI\frac{dy}{dx} = -\frac{25x^3}{6} + \frac{125x^2}{2} - 1041\tfrac{2}{3}, \tag{3}$$

from which by substituting the abscissa of any section of the beam for x and dividing the result by EI we get the tangent of the angle of slope for that section. Substituting the value of C_1, in equation (2) gives

$$EIy = -\frac{25x^4}{24} + \frac{125x^3}{6} - 1041\tfrac{2}{3}\,x, \qquad (4)$$

from which the ordinate of the neutral line (curve of bending) at any section may be found. In finding the value of this ordinate, or of the slope, care must be taken to use the same units throughout; for example, if we have a steel beam and E is given as 30,000,000, it is in lbs. per sq. in.; we must therefore use x in ins., find I for the section in in. units, and reduce the bending moment to in.-lbs.; this will give us the deflection in ins.

It is convenient to solve this problem by using letters to represent the numerical data and to make the substitutions later in the equation which gives the desired result; for example, in the above problem let the load per ft.-run equal "w" and the length of the beam "a," then

$$L = -w,$$

$$F = -wx + C = -wx + P = -wx + \frac{wa}{2},$$

$$M = -\frac{wx^2}{2} + \frac{wax}{2} + [C_1 = 0,$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wax^2}{4} + C_2.$$

$$0 = -\frac{w\left(\frac{a}{2}\right)^3}{6} + \frac{wa\left(\frac{a}{2}\right)^2}{4} + C_2. \quad \therefore C_2 = -\frac{wa^3}{24},$$

$$EIy = -\frac{wx^4}{24} + \frac{wax^3}{12} - \frac{wa^3x}{24} + [C_3 = 0.$$

By this latter method, the particular result desired may be obtained without doing the numerical work required for the

other equations. In the above beam the maximum slope is at the ends, and the tangent of the angle is $-\frac{wa^3}{24EI}$. The maximum deflection occurs at the middle of the beam, and there $y = -\frac{5wa^4}{384EI}$.

46. A beam, supported at the ends, carries a single concentrated load W at a distance "a" from the left end and "b" from the right end. Find equation for slope and deflection.

In this problem the beam will be bent as shown in Fig. 37, and the elastic curve will consist of two branches, the part

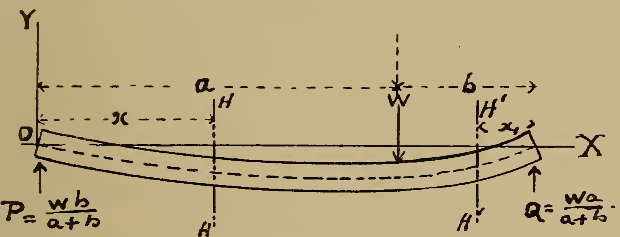

FIG. 37.

from O to the weight and that from the weight to the other end of the beam. The shearing force on each part will be constant, but the values will be different and they will have different signs. We must consider the two parts separately. With the origin at O, the bending moment for any section HH to the *left* of the load W is

$$M = \frac{Wbx}{a+b},$$

$$EI\frac{dy}{dx} = \frac{Wbx^2}{2(a+b)} + C_1,$$

$$EIy = \frac{Wbx^3}{6(a+b)} + C_1x + [C_2 = 0.$$

For the part to the right of the load W we will take our origin at the right end, then, letting x_1 be the distance from

the right end to any section $H'H'$, the bending moment will be

$$M = -Qx_1 = -\frac{Wax_1}{a+b},$$

the sign being negative because the moment tends to turn the right end of the beam counter clockwise, or in the opposite direction to that of the other end; we have then for the *right* end

$$M = -\frac{Wax_1}{a+b},$$

$$EI\frac{dy}{dx} = -\frac{Wax_1^2}{2(a+b)} + C'_1.$$

$$EIy = -\frac{Wax_1^3}{6(a+b)} + C'_1x_1 + [C'_2 = 0.$$

If we move the origin for these equations back to the left end of the beam, by substituting $\{x - (a+b)\}$ for x_1, we get

$$M = \frac{Wa(a+b-x)}{a+b},$$

$$EI\frac{dy}{dx} = -\frac{Wa(a+b-x)^2}{2(a+b)} + C'_1,$$

$$EIy = \frac{Wa(a+b-x)^3}{6(a+b)} - C'_1(a+b-x).$$

Now for the section under W the slope and deflection is the same using either the equations for the part of the curve to the left of W, or those for the part to the right of W, so we can put the corresponding values equal to each other when $x = a$,

$$\frac{Wba^2}{2(a+b)} + C_1 = \frac{Wab^2}{2(a+b)} + C'_1,$$

$$\text{or, } C_1 - C'_1 = \frac{Wab}{2}. \qquad (1)$$

$$\frac{Wba^3}{6(a+b)} + C_1a = \frac{Wab^3}{6(a+b)} - C'_1b,$$

$$\text{or, } C_1a + C'_1b = \frac{Wab(b-a)}{6}. \qquad (2)$$

From the simultaneous equations (1) and (2) we get

$$C_1 = -\frac{Wab(a+2b)}{6(a+b)}, \text{ and } C'_1 = \frac{Wab(2a+b)}{6(a+b)}$$

and substituting these values of C_1 and C'_1 we get

for the left end,

$$EI\frac{dy}{dx} = \frac{Wbx^2}{2(a+b)} - \frac{Wab(a+2b)}{6(a+b)}$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wab(a+2b)x}{6(a+b)}$$

for the right end,

$$EI\frac{dy}{dx} = -\frac{Wa(a+b-x)^2}{2(a+b)} + \frac{Wab(2a+b)}{6(a+b)}$$

$$EIy = \frac{Wa(a+b-x)^3}{6(a+b)} - \frac{Wab(2a+b)(a+b-x)}{6(a+b)}$$

and from either set of these equations we get the deflection *at the load* to be

$$y = -\frac{Wa^2b^2}{3EI(a+b)}.$$

Since the load is not at the middle of the beam, the maximum deflection will occur in the longer segment, and at the section of maximum deflection the slope will be zero; therefore, supposing "a" to be greater than "b," if we put the equation of the *slope* for the part of the curve to the *left* of W equal to zero and solve for x we will get the abscissa of the section of maximum deflection; and substituting this value of x in the equation for the deflection will give us the maximum deflection, as follows:

$$0 = \frac{Wbx^2}{2(a+b)} - \frac{Wab(a+2b)}{6(a+b)} \text{ gives } x = \sqrt{\frac{a(a+2b)}{3}},$$

and this value of x substituted in

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wab(a+2b)x}{6(a+b)}$$

shows the maximum deflection to be

$$y_{max} = -\frac{Wb}{3EI(a+b)}\left(\frac{a(a+2b)}{3}\right)^{\frac{3}{2}},$$

or, if $a = b$,

$$y_{max} = -\frac{Wa^3}{6EI}.$$

Examples:

1. A steel beam, 10 ft. long, supports a concentrated load of 25 tons at its middle. What is the deflection under the load if I in in. units is 84.9 and $E = 30,000,000$? Find equation of elastic curve.

Ans. .077 — ins.

2. A steel beam of I section, 20 ft. long, 8 ins. deep, 3 ins. wide, with flanges and web each $\frac{1}{2}$ in. thick, is used to support a floor weighing 200 lbs. per sq. ft. The beams are supported at the ends and spaced 3 ft. apart. What is the maximum deflection? $E = 30,000,000$. Find equation of elastic curve.

Ans. 1.3 — ins.

3. A beam, supported at the ends, carries a uniform load of 150 lbs. per ft.-run. It is 10 ft. long, 6 ins. deep, and 4 ins. wide. $E = 1,200,000$. Find equation of elastic curve and the maximum deflection.

Ans. .51 — ins.

4. A beam 12 ft. long, 8 in. deep, and 12 in. wide, is supported at the ends and carries a load of 1200 lbs., 3 ft. from the left end. $E = 1,200,000$. Find the elastic curve and the maximum deflection.

Ans. .12 + ins.

5. A steel beam, 60 ft. long and weighing 100 lbs. per ft., carries a concentrated load of 14,400 lbs. at the middle. If I in in. units is 1160, find the maximum deflection. $E = 30,000,000$.

Ans. 6.38 — ins.

6. A beam, 20 ft. long, supports a floor weighing 100 lbs. per sq. ft. The beams are spaced 4 ft. apart. Supposing the section of the beam to be square, find the side of the square in order that the deflection may not exceed $\frac{1}{2}$ in. $E = 700$ in.-tons.

Ans. Side is 12.18 ins.

CHAPTER IX.

SLOPE AND DEFLECTION—Continued.

47. Example: A beam of length "a" is *fixed* at the left end and unsupported at the right. It is loaded with a uniform load of w lbs. per ft.-run. Find the equation of the elastic curve.

$$L = -w,$$

$$F = -wx + C = -wx + wa \quad (P = \text{whole load}),$$

$$M = -\frac{wx^2}{2} + wax + \left[C_1 = -\frac{wa^2}{2} \right. (C_1 \text{ is found in Art. 40}),$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wax^2}{2} - \frac{wa^2x}{2} + [C_2 = 0 \quad \text{(beam \textit{fixed} at origin)},$$

$$EIy = -\frac{wx^4}{24} + \frac{wax^3}{6} - \frac{wa^2x^2}{4} + [C_3 = 0 \quad \text{(beam \textit{fixed} at origin)}.$$

The maximum deflection obviously occurs where $x = a$.

$$\therefore y_{max} = -\frac{wa^4}{8EI}.$$

If we suppose the free end of the above beam is propped up to the same level as the fixed end. The prop now supports some of the load so we do *not* know P.

$$L = -w,$$

$$F = -wx + C = -wx + P,$$

$$M = -\frac{wx^2}{2} + Px + C_1.$$

We know the bending moment at the end *supported* by the prop is zero, so putting $M = 0$ and substituting $x = a$

$$0 = -\frac{wa^2}{2} + Pa + C_1,$$

from which

$$C_1 = \frac{wa^2}{2} - Pa.$$

Substituting this value of C_1 we get

$$M = -\frac{wx^2}{2} + Px + \frac{wa^2}{2} - Pa,$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{Px^2}{2} + \frac{wa^2x}{2} - Pax + [C_2 = 0$$

(beam *fixed* at origin),

$$EIy = -\frac{wx^4}{24} + \frac{Px^3}{6} + \frac{wa^2x^2}{4} - \frac{Pax^2}{2} + [C_3 = 0$$

(beam *fixed* at origin).

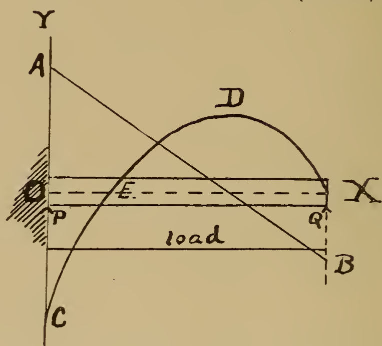

FIG. 38.

We can now Find P, for the deflection is zero where $x = a$, and all the other constants are determined.

$$0 = -\frac{wa^4}{24} + \frac{Pa^3}{6} + \frac{wa^4}{4} - \frac{Pa^3}{2},$$

from which

$$P = \frac{5wa}{8},$$

and substituting this value in the equations will give us the curves desired and the slope and deflection. The curves are shown plotted in Fig. 38. AB being the curve of S. F., and CDX that of bending moment. The B. M. at the origin is *negative* and equal to $-\frac{wa^2}{8}$. The greatest *positive* B. M.

occurs where the shearing force is zero at $x = \frac{5a}{8}$, and substituting this value in the equation for B. M. gives the greatest positive B. M. equal to $\frac{9wa^2}{128}$. The maximum B. M. is therefore the negative one, and the beam will break if overloaded, at the *fixed* end.

It will be noticed that the B. M. is zero at the free end, and also at a point between there and the fixed end. Putting the equation for B. M. equal to zero and solving for x, gives $x = a$ and $x = \frac{a}{4}$. The point where $x = \frac{a}{4}$ is called a *virtual joint, because,* if a beam loaded and supported in this way had a hinge at this point, the bending moment would not cause it to turn. Putting $\frac{dy}{dx} = 0$, and solving for x gives $x = .58a$ and a value *greater* than the length of the beam. Where $x = .58a$ the slope is zero, and this is the point of maximum deflection. Substituting this value of x in the equation for deflection gives

$$y_{max} = -\frac{.006wa^4}{EI}.$$

Compare the curve of B. M. for this beam with that in Fig. 33. As soon as the prop under the free end of this beam begins to bear a part of the load there exists a positive B. M. and the *virtual joint* which is then near the right end of the beam, moves toward the *fixed* end as the right end is propped up and the load on the prop increases; but the results due to change in conditions are readily followed.

48. A beam "a" ft. long is fixed at both ends and carries a concentrated load W at its middle. Here $P = Q = \frac{W}{2}$.

$\therefore$ S. F. $= \frac{W}{2}$ and is constant from the ends to the load, but has opposite directions on opposite sides of the load; there-

fore, as in Art. 44, we must consider the two parts of the beam separately. To the left of the load, then,

$$M = \frac{Wx}{2} + C. \quad (C \text{ is not known}),$$

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} + Cx + [C_1 = 0 \text{ (beam fixed at end)}.$$

The loading being symmetrical, the slope at the middle is zero, substituting $x = \frac{a}{2}$,

$$0 = \frac{W\left(\frac{a}{2}\right)^2}{4} + C\left(\frac{a}{2}\right); \text{ or, } C = -\frac{Wa}{8}.$$

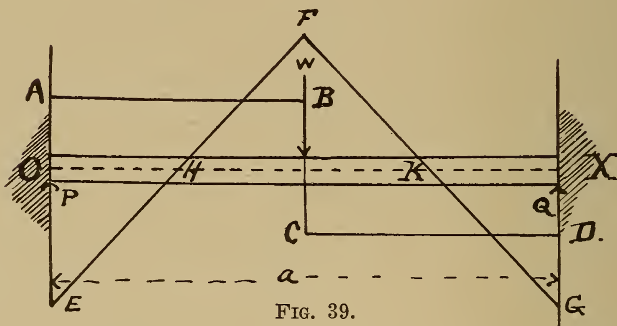

FIG. 39.

The equation of the curve of bending to the left of the load, then, is

$$M = \frac{Wx}{2} - \frac{Wa}{8}.$$

This is an equation of the first degree, and therefore represents a straight line (EF in the figure). We know the B. M. is symmetrical, and that the curve for the right end will have equal ordinates, but of opposite sign (the curve being FG in the figure). Now it is clear we cannot embrace both of these straight lines in one equation of the first degree, and for this reason we must consider the two parts separately. Before

proceeding, we will notice that the bending moment has the same value with different signs at the ends and in the middle. This shows that the beam is as likely to break at the ends as in the middle if overloaded, and also that the virtual joints are at quarter span.

$$M = \frac{Wx}{2} - \frac{Wa}{8},$$

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wax}{8} + [C_1 = 0,$$

$$EIy = \frac{Wx^3}{12} - \frac{Wax^2}{16} + [C_2 = 0.$$

The maximum deflection is at the middle and equal to

$$y_{max} = \frac{Wa^3}{24EI}.$$

49. The Cantilever Bridge.—In the cantilever bridge the joints are placed at the points where the bending moment would be zero *if the bridge were continuous over the whole span.* We will consider a beam *fixed* at *both* ends, having two fixed joints and carrying a uniform load of w lbs. per ft.-run. There being joints at F and G (Fig. 40) we must consider the part FG as a separate beam supported at the ends, the load on it being equally divided and supported at F and G by the beams OF and GX. From the conditions we could assume P and Q *with this loading* to be equal to $\frac{w(a+b+c)}{2}$, but we can get their values in another way:

$$L = -w,$$
$$F = -wx + C = -wx + P,$$
$$M = -\frac{wx}{2} + Px + C_1.$$

The bending moment at the joint is zero, so

$$0 = -\frac{wa^2}{2} + Pa + C_1, \qquad (1)$$

$$0 = -\frac{w(a+b)}{2} + P(a+b) + C_1, \qquad (2)$$

and from equations (1) and (2) we get

$$P=\frac{w}{2}(2a+b), \text{ and } C_1=-\frac{wa}{2}(a+b),$$

substituting,

$$F=-wx+\frac{w}{2}(2a+b),$$

$$M=-\frac{wx^2}{2}+\frac{w}{2}(2a+b)x-\frac{wa}{2}(a+b).$$

The joints being where B. M. is zero, the curves of S. F. and B. M. will be continuous as shown in Fig. 40, but for slope and deflection we must consider each part as a separate beam,

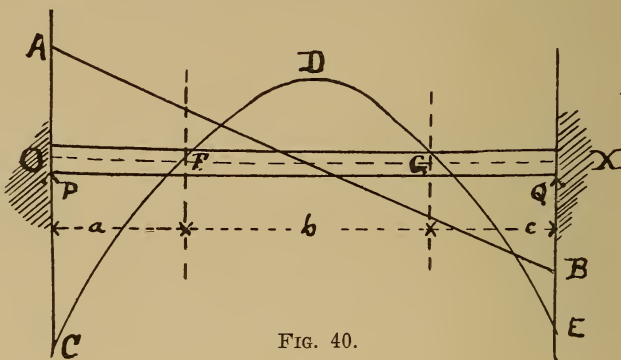

FIG. 40.

the parts at ends having in addition to their uniform load a concentrated load at their ends, F and G, equal to half the total load on the middle part. For equal strength, the bending moments at the ends and middle of the above beam should have equal values and this requires "a" to be equal to "c," and

$$\frac{wa}{2}(a+b)=\frac{wb^2}{8},$$

whence

$$4a^2+4ba=b^2;\text{ or, }4a^2+4ba+b^2=2b^2;\text{ or, }(2a+b)^2=2b^2;$$

or the square of the whole span must equal $2b^2$, and the joint must be distant from the middle of the beam $\frac{span}{2\sqrt{2}}$.

50. Travelling Loads.—If a load W moves from left to right across a beam of length "a," the positive shearing force for any position of W is equal to P (Fig. 41). P is obviously greatest when W is *just* over it, and as W moves across the beam, P's value drops uniformly until when W is just over Q, P's value is zero. We can then represent the *change* of the positive S. F. by the line AX, and by the same reasoning the change in the negative S. F. would be represented by OB. Considering the B. M. we know that for any position of W the B. M. is greatest for the section under W and

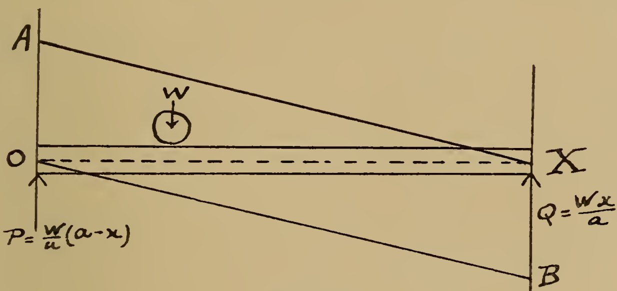

FIG. 41.

for that section is $M = Px = \frac{W}{a}(a - x)x$. By the principles of maxima and minima, we can find the value of x for which M is a maximum by putting the first derivative of M with respect to x equal to zero and solving for x, as

$$\frac{dM}{dx} = \frac{W}{a}(a - 2x) = 0,$$

which gives

$$x = \frac{a}{2},$$

the value of x, or the position of W which gives the maximum B. M. If the travelling load is continuous, such as a train

of cars crossing a bridge, the maximum value occurs when, if the train is long enough, it extends completely over the bridge; if it is not long enough for this the maximum positive S. F. occurs when the whole train has just got on to the bridge and the maximum B. M. when the middle of the train is just over the middle of the bridge.

51. Oblique Loading.—Loading is said to be oblique when, for any cross-section of a beam, the plane of the external bending moment does not pass through a principal axis of the section.

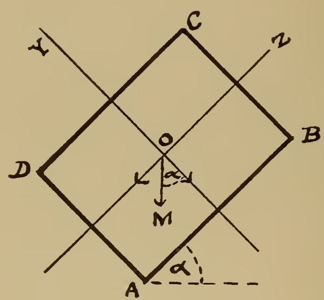

FIG. 42.

Let Fig. 42 represent the section of a beam and let the arrow-head marked M represent the *direction* of the B. M. The axis of X is perpendicular to the plane of the paper through O, and the axes of Y and Z are shown. Resolve M into components parallel to the axes of Y and Z. The direction of the fiber stress due to bending is *perpendicular to the plane of the paper,* and that part of it, due to $M \sin \alpha$, which acts normal to the side AD of the section is

$$p_z = \frac{M \sin \alpha \cdot z}{I_y},$$

which is obtained by substituting $M \sin \alpha$, the component moment, in $\frac{p}{y} = \frac{M}{I}$, the general equation for bending, I_y being the moment of inertia of the section about the axis of Y, and z the distance of the line AD from the axis of Y. The normal fiber stress perpendicular to the plane of the paper on AB is

$$p_y = \frac{M \cos \alpha}{I_z} \cdot y.$$

Applying the principle of superposition the total fiber stress at A would be

$$p = p_y + p_z = \frac{M \sin \alpha \, . \, z}{I_y} + \frac{M \cos \alpha \, . \, y}{I_z}.$$

52. The *work* done in bending a beam is evidently equal to $M\theta$, where M is a *uniform* bending moment and θ is the angle of slope *at the ends*. If l is the length of the beam and R the radius of the curve into which it is bent, then $\theta = \frac{l}{2R}$, and from $\frac{M}{I} = \frac{E}{R}$ we have $R = \frac{EI}{M}$. Hence the work done by a *uniform* bending moment is equal to $\frac{M^2 l}{2EI}$. But bending moments are seldom uniform, so, for a variable bending moment

$$\text{Work} = \frac{1}{2EI} \int M^2 dx.$$

Examples:

1. A dam is supported by a row of uprights *fixed* at their base and their upper ends are held vertical by struts sloping at 45°. Water pressure varies as the depth, or $L = -wx$. Find the equation for deflection. Length of upright "a." What is the thrust on the struts?

Ans. Thrust $= \frac{2}{7}$ horizontal pressure of water.

2. A railroad is inclined at 30° to the horizontal. The stringers are 10.5 ft. apart and the rails are 1 ft. inside the stringers. The ties are 8 in. deep and 6 in. wide. The load transmitted by each rail to one tie is 10 tons. What is the maximum normal stress in each tie?

Ans. 5744 lb.-ins.

3. A beam, $2a$ ft. long, *fixed* at the ends is uniformly loaded with w lbs. per ft.-run. Find the maximum deflection and the virtual joints.

Ans. Maximum deflection $= -\frac{Wa^4}{24EI}$. Virtual joints where $x = a(1 \pm \sqrt{\frac{1}{3}})$.

4. A beam, "a" ft. long and *fixed* at the ends, carries a uniformly increasing load from zero at the left end to w lbs. per ft.-run at the right. Find the maximum deflection and the virtual joints.

Ans. Maximum deflection $= -\frac{4Wa^4}{3125EI}$. Virtual joints where $x = \frac{a}{2}$ and $\frac{2a}{5}$.

5. A beam, "a" ft. long, *fixed* at one end and the other end propped up to the same level, carries a uniformly decreasing load from w lbs. per ft.-run at the *fixed* end, to zero at the propped end. Find the equation for deflection, the position of the maximum positive B. M., and the position of the virtual joints.

Ans. Maximum positive B. M. where $x = a\sqrt{\frac{1}{5}}$. Virtual joints where $x = a\sqrt{\frac{3}{5}}$.

6. A beam, "a" ft. long, fixed at both ends, carries a single concentrated load W at a distance "d" from the left end. Find the deflection under the load.

Ans. $\delta = \frac{Wd^3(a-d)^3}{3EIa^3}$.

7. If the load of the beam in example 6 had been at the middle of the beam, what would have been the maximum deflection?

Ans. $\delta = \dfrac{Wa^3}{192EI}$.

8. A single load of 50 tons crosses a bridge of 100-ft. span. Draw the curves of maximum S. F. and B. M., and give the values of those quantities at half and quarter span.

9. A train, weighing 1 ton per ft.-run and 112 ft. long, crosses a bridge of 100-ft. span. Draw curves of maximum S. F. and B. M., and give values at half and quarter span.

10. A steel shaft carries a load equal to k times its own weight, first uniformly distributed, second concentrated at its middle; considering it as a beam *fixed* at the ends, find the distance apart of the bearings that the ratio of deflection to span may be $\frac{1}{1200}$.

Ans. (1) Span in ft. $= 10.5 \sqrt[3]{\dfrac{d^2}{k+1}}$.

(2) Span $= 8.3 \sqrt[3]{\dfrac{d^2}{k+\frac{1}{2}}}$ (d in ins.).

CHAPTER X.

CONTINUOUS BEAMS.

53. A continuous beam is one which extends over several supports. In this chapter we will consider as heretofore only such beams as have a uniform cross-section and, as in all practical constructions the supports are adjusted so as to allow as little strain as possible in the material, we will assume all the supports to be at the same level. It is obvious

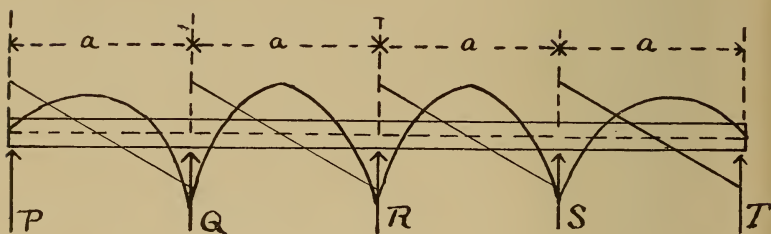

FIG. 43.

that the bending moments at the intermediate supports will be negative as at these points the conditions are similar to those of a beam balanced over a single support. If the end spans are short we may have the supporting forces at the ends equal to zero, and if the ends are "anchored" (fastened down to the support) we may have a negative supporting force, that is one acting in the same direction as the loads. In Fig. 43, we see approximately the curves of bending moments and shearing force for a continuous beam carrying a uniform load and having five equally spaced supports. There are two virtual joints between supports except for the end

spans, and we will find that the value of the deflection in each span is somewhere between that for a uniformly loaded beam of span length *supported* at the ends, and one uniformly loaded of span length *fixed* at the ends. The difficulty in working with continuous beams lies in finding the supporting forces, for the usual method of taking moments will not do, there being two unknown quantities in each equation, and the equations reducing to identities when we attempt to solve them. For beams having only *three* supporting forces, the solutions are not difficult, as will be shown; but as the number of supports increases the calculations become more and more tedious.

54. A beam of length $2a$, carries a uniform load and is supported by three equidistant supports. A beam of length $2a$ carrying a uniform load of w per unit length and supported at the ends, has a maximum deflection equal to $-\frac{5w(2a)^4}{384EI}$ (see Art. 45). A like beam supported at the ends and carrying a concentrated load R at its middle has a maximum deflection equal to $-\frac{R(2a)^3}{48EI}$ (Art. 46). Now the above continuous beam may be considered as a uniformly loaded beam which has a prop at its middle by which the middle point is propped up to the same level as the end supports. Obviously then the deflection at any point will be that due to the uniform load minus that due to the thrust of the prop, and as the prop deflects upward we have

$$0 = \frac{R(2a)^3}{48EI} - \frac{5w(2a)^4}{384EI},$$

or R, the thrust of the prop, or the middle supporting force is

$$R = \tfrac{5}{4}\, wa.$$

The supporting forces at the ends are obviously equal, so

$$P = Q = \frac{w(2a) - \frac{5}{4}wa}{2} = \tfrac{3}{8}wa.$$

Having the supporting forces we have now no difficulty in getting the equations for slope and deflection, for taking the origin at the middle; the B. M. for any section distant x from the origin is, by definition (remembering that the part of R which is due to the loading on one side of the origin is $\frac{R}{2}$).

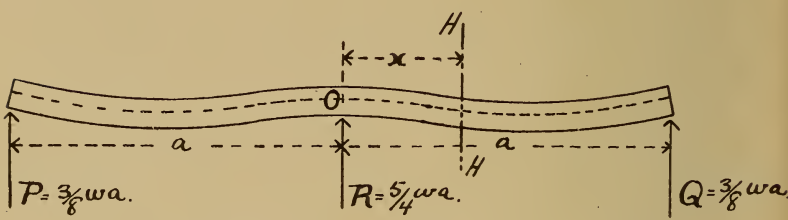

FIG. 44.

$$M = EI\frac{d^2y}{dx^2} = \frac{R}{2}x - wx \cdot \frac{x}{2} + M_0.$$

$$EI\frac{dy}{dx} = \frac{Rx^2}{4} - \frac{wx^3}{6} + M_0x + [C = 0 \ldots \left\{\begin{array}{l}\text{from symmetry}\\ \text{the slope is zero}\\ \text{at the origin.}\end{array}\right.$$

$$EIy = \frac{Rx^3}{12} - \frac{wx^4}{24} + M_0\frac{x^2}{2} + [C_1 = 0 \ldots \left\{\begin{array}{l}\text{deflection is zero}\\ \text{at the origin.}\end{array}\right.$$

Substituting the value of R we have for the B. M. at any section

$$M = \tfrac{5}{8}wax - \frac{wx^2}{2} + M_0.$$

And as the B. M. is zero at the ends, we have, substituting $x = a$, the value of M_0 equal to $-\frac{wa^2}{8}$; or

$$M = \tfrac{5}{8}wax - \frac{wx^2}{2} - \frac{wa^2}{8};$$

putting this value of M equal to zero and solving for x gives us $x = \pm \frac{a}{4}$ as the position of the virtual joints. Integrating we get the equations for slope and deflection

$$EI\,\frac{dy}{dx} = \frac{5}{16}\,wax^2 - \frac{wx^3}{6} - \frac{wa^2x}{8} + [C = 0,$$

and

$$EIy = \frac{5}{48}\,wax^3 - \frac{wx^4}{24} - \frac{wa^2x^2}{16} + [C_1 = 0.$$

To show how carefully the level of the supports must be adjusted, suppose there is left a deflection over the middle support, equal, let us say, to $\frac{1}{5}$ of the total deflection which would occur were there no middle support; then

$$\tfrac{1}{5}\cdot\frac{5w(2a)^4}{384EI} = \frac{5w(2a)^4}{384EI} - \frac{R(2a)^3}{48EI},$$

from which $R = \frac{w(2a)}{2}$, or one-half the total load. If this deflection were in the other direction, or $-\frac{1}{5}$ we would have $R = \frac{3}{4}$ the total load. Remembering that the deflection in any case is very small when we see that a variation of it $\frac{1}{5}$ up or down causes R to change from $\frac{3}{4}$ to $\frac{1}{2}$ the total load, we can understand how carefully the supports must be adjusted to the same level.

55. With concentrated loads we must proceed in a different way. Suppose a beam which is supported at the middle and ends carries a concentrated load W midway between the supports, as represented in Fig. 45. Taking the origin at the middle as before and considering the right half of the beam, the bending moment at any section HH between the origin and the load is (Art. 46),

$$M = EI\,\frac{d^2y}{dx^2} = -\,W\left(\frac{a}{2} - x\right) + Q\,(a - x), \qquad (1)$$

Integrating,

$$EI\,\frac{dy}{dx} = -\frac{Wax}{2} + \frac{Wx^2}{2} + Qax - \frac{Qx^2}{2} + [C = 0, \qquad (2)$$

and

$$EIy = -\frac{Wax^2}{4} + \frac{Wx^3}{6} + \frac{Qax^2}{2} - \frac{Qx^3}{6} + [C_1 = 0. \qquad (3)$$

For a section between W and the end of the beam

$$M = EI\frac{d^2y}{dx^2} = Q(a - x). \qquad (4)$$

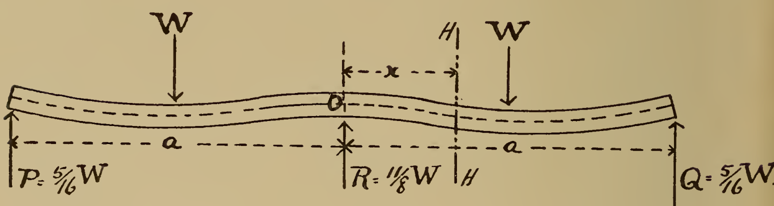

FIG. 45.

Integrating,

$$EI\,\frac{dy}{dx} = Qax - \frac{Qx^2}{2} + [C = ? \qquad (5a)$$

The slope under the load is the same in both branches of the curve, therefore, substituting $x = \frac{a}{2}$ in equations (2) and (5a), and equating them we have

$$-\frac{Wa^2}{4} + \frac{Wa^2}{8} + \frac{Qa^2}{2} - \frac{Qa^2}{8} = \frac{Qa^2}{2} - \frac{Qa^2}{8} + C, \text{ or } C = -\frac{Wa^2}{8}.$$

Substituting,

$$EI\,\frac{dy}{dx} = Qax - \frac{Qx^2}{2} - \frac{Wa^2}{8}. \qquad (5)$$

Integrating,

$$EIy = \frac{Qax^2}{2} - \frac{Qx^3}{6} - \frac{Wa^2x}{8} + [C_1 = ?. \qquad (6a)$$

The deflection under the load is the same in both branches, so as before, using equations (3) and (6a) we have

$$-\frac{Wa^3}{16}+\frac{Wa^3}{48}+\frac{Qa^3}{8}-\frac{Qa^3}{48}=\frac{Qa^3}{8}-\frac{Qa^3}{48}-\frac{Wa^3}{16}+C_1;$$

or,

$$C_1=\frac{Wa^3}{48}.$$

Substituting,

$$EIy=\frac{Qax^2}{2}-\frac{Qx^3}{6}-\frac{Wa^2x}{8}+\frac{Wa^3}{48}. \qquad (6)$$

Now the deflection at the right support is equal to zero, and putting equation (6) equal to zero, substituting $x=a$, solving for Q, gives $Q=\frac{5}{16}W$. From symmetry, P equals Q, and R must support the rest of the load, hence

$$R=2W-(P+Q)=\tfrac{11}{8}W.$$

The methods of this and the preceding article will cover most of the cases of continuous beams found in actual practice, but there are many cases where beams have more than three supports, and for these a method known as the "Method of Three Moments" must be employed. This method came into use about 1857, and is a general method for obtaining the supporting forces.

56. Theorem of Three Moments.—Let Fig. 46 represent the part of a beam over any three *consecutive* supports, and let the loading be uniform between adjacent supports. Take the origin at P and let the bending moments at P, R and Q be M_1, M_2 and M_3 respectively, the distance between supports to be a and a_1, the loading between P and R be w lbs. per unit length, and between R and Q, w_1 lbs. per unit length; then the bending moment at any section HH between P and R is (Art. 45),

$$M=-\frac{wx^2}{2}+Px+C,$$

but C in this case is equal to M_1, so

$$M = EI\frac{d^2y}{dx^2} = M_1 + Px - \frac{wx^2}{2}. \tag{1}$$

Integrating,

$$EI\frac{dy}{dx} = M_1 x + P\frac{x^2}{2} - \frac{wx^3}{6} + [C = ?, \tag{2}$$

and

$$EIy = M_1\frac{x^2}{2} + P\frac{x^3}{6} - \frac{wx^4}{24} + Cx + [C_1 = 0$$

(level supports). (3)

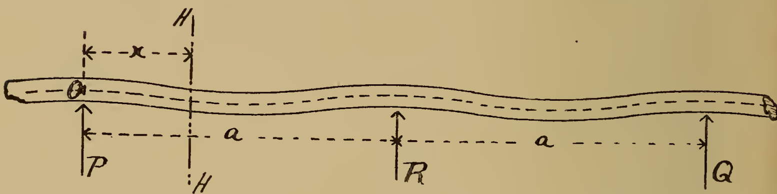

FIG. 46.

Now $y = 0$ when $x = a$, hence

$$0 = \frac{M_1a^2}{2} + \frac{Pa^3}{6} - \frac{wa^4}{24} + Ca. \quad \therefore C = -\frac{M_1a}{2} - \frac{Pa^2}{6} + \frac{wa^3}{24}.$$

Substituting,

$$EI\frac{dy}{dx} = M_1x + \frac{Px^2}{2} - \frac{wx^3}{6} - \frac{M_1a}{2} - \frac{Pa^2}{6} + \frac{wa^3}{24}, \tag{4}$$

and

$$EIy = \frac{M_1x^2}{2} + \frac{Px^3}{6} - \frac{wx^4}{24} - \frac{M_1ax}{2} - \frac{Pa^2x}{6} + \frac{wa^3x}{24}. \tag{5}$$

From (1), if we let $x = a$, we get M_2 in terms of M_1, or,

$$M_2 = M_1 + Pa - \frac{wa^2}{2}. \tag{6}$$

Now if we imagine the beam reversed and use Q for the origin, we can in the same way find the equation for bending moment, slope and deflection; for the part of the beam be-

tween Q and R, and over R the values of the bending moment, slope and deflection will be the same for both branches of the curve. We can therefore equate these values if we substitute $x = a$ in those of the left branch and $x = a_1$ in those of the right. Making these substitutions and equating we get

for the left branch, for the right branch,

$$M_1 + Pa - \frac{wa^2}{2} = \quad M_2, \quad = M_3 + Qa_1 - \frac{w_1 a_1}{2}, \qquad (7)$$

$$\frac{M_1 a}{2} + \frac{Pa^2}{3} - \frac{wa^3}{8} = \text{slope over } R \quad = -\frac{M_3 a_1}{2} - \frac{Qa_1^2}{3} + \frac{w_1 a_1^3}{8}. \qquad (8)$$

If we substitute the values of Q and P from equations (7) in equation (8) and reduce we get

$$M_1 a + 2M_2(a + a_1) + M_3 a_1 = -\frac{wa^3 + w_1 a_1^3}{4}. \qquad (9)$$

This is known as the equation of the three moments, and by using different sets of three consecutive supports we can get as many equations, *less two*, as there are supports. From our knowledge of conditions *at the ends* of the continuous beam we can get two more equations involving the moments over the supports, and thus having as many equations as there are unknown quantities we may find all the supporting forces.

For example, let the beam of Fig. 43 be supported at equal intervals and uniformly loaded with w lbs. per unit length. From equation (9)

$$M_1 a + 4M_2 a + M_3 a = -\frac{wa^3}{2},$$

or,

$$M_1 + 4M_2 + M_3 = -\frac{wa^2}{2}. \qquad (1)$$

$$M_2 + 4M_3 + M_4 = -\frac{wa^2}{2}. \qquad (2)$$

$$M_3 + 4M_4 + M_5 = -\frac{wa^2}{2}. \qquad (3)$$

The bending moment is zero at the ends, hence $M_1 = M_5 = 0$. From (1) and (3)

$$M_2 = M_4.$$

Substituting, we get from (2) and (1)

$$M_2 = -\frac{3wa^2}{28}.$$

From (6), using the first three supports,

$$M_2 = M_1 + Pa - \frac{wa^2}{2} = -\frac{3wa^2}{28}. \quad \therefore P = \tfrac{11}{28} wa.$$

In the same way $Q = \frac{32}{28} wa$, and from symmetry we know $S = Q$ and $T = P$, so the total load minus $P + Q + T + S = R$ from which $R = \frac{26}{28} wa$.

Examples:

1. A continuous beam of five equal spans is uniformly loaded with w lbs. per ft.-run. If the beam is 38 ft. long, what are the supporting forces and the bending moments at the supports?

2. Find the side of a steel beam of square section to span four openings of 8 ft. each, the total load per span being 44,000 lbs. and the greatest horizontal fiber stress not to exceed 15,000 lbs. per sq. in.

3. Find the depth of a steel beam of rectangular section twice as deep as broad, to span three openings each 12 ft. wide, the total load on each span being 6000 lbs. and the greatest fiber stress allowed being 12,000 lbs. per sq. in.

4. A continuous beam of three spans is loaded only on the middle span with a uniform load. What are the three supporting forces?

Ans. At ends $-\frac{wl}{20}$.

5. A steel beam, $I = 268.9$ in in. units, is 36 ft. long and spans four openings, the end spans being each 8 ft., and the middle ones each 10 ft. Find the maximum B. M. of the beam. What will be the uniform load per ft. to make a maximum fiber stress of 15,000 lbs. per sq. in.?

6. A continuous beam carries a uniform load. If the end spans are each 80 ft. what will be the length of the middle span in order that the B. M. at its middle may equal zero?

7. How much work is done on a beam 10 ft. long, 10 ins. deep and 8 ins. wide, which carries a uniform load of 250 lbs. per ft.-run?

CHAPTER XI.

COLUMNS AND STRUTS.

57. When a prismatic piece of material, several times longer than its greatest breadth, is under compression, it is called a *column* or *strut;* a column being such a piece placed *vertically* and carrying a static load, and all others being struts.

A column or strut of material which is not homogeneous, or one on which the load does not act exactly in the geometric axis, will bend or buckle. For mathematical investigation we must assume our material homogeneous, and instead of assuming a slight deviation of the line of action of our load from the geometric axis, we will assume that the column has first been bent by a horizontal force, then such a load applied to its end as will just keep it in the bent form after the removal of the horizontal force. The assumptions we make will be three: (1), the column perfectly straight originally; (2), material perfectly homogeneous; and (3), load applied exactly over the center of the ends. These conditions are never *exactly* fulfilled in practice, so a rather large factor of safety must be applied.

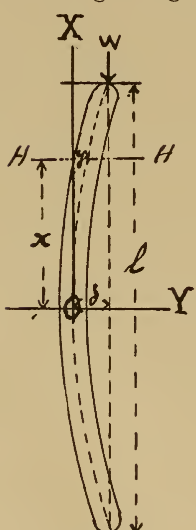

FIG. 47.

58. Euler's Formula for Long Columns.—We will choose a column with rounded or pivoted ends, so that they will be free to move slightly as the column is bent.

Let Fig. 47 represent such a column, held in the bent position by the load W. Let the origin be at the center of the column, the axis of X vertical, and that of Y horizontal; δ is the maximum deflection, and y the ordinate of the neutral line at any section HH distant x from the origin. From Art. 43 we have the general equation of bending

$$EI\frac{d^2y}{dx^2} = M.$$

The bending moment for the section HH is, from the figure,

$$M = W(\delta - y),$$

so

$$\frac{d^2y}{dx^2} = \frac{W}{EI}(\delta - y),$$

multiplying both members of this equation by $2\frac{dy}{dx}$ and integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{2W\delta y}{EI} - \frac{Wy^2}{EI} + [C_1 = 0; \qquad (1)$$

now y is zero at the origin and $\frac{dy}{dx}$ is also zero there, the tangent to the curve of bending being parallel to the axis of X at that point, $\therefore$ the constant of integration being this tangent is zero.

Extracting the square root of both members of equation (1) and, transposing, we get

$$\frac{1}{\delta} \cdot \frac{dy}{\sqrt{1 - \left(\frac{\delta - y}{\delta}\right)^2}} = \sqrt{\frac{W}{EI}} \cdot dx,$$

which, integrated, gives

$$\cos^{-1}\left(\frac{\delta - y}{\delta}\right) = \sqrt{\frac{W}{EI}}\,x + [C_2 = 0. \qquad (2)$$

C_2 is zero, for where $x = 0$, $y = 0$ and $\cos^{-1} 1 = 0$.

Letting l equal the length of the column, when $x = \frac{l}{2}$, we have $y = \delta$, substituting these values in equation (2) we get

$$\cos^{-1} 0 = \sqrt{\frac{W}{EI}} \cdot \frac{l}{2}.$$

The angle whose cosine is zero is $\frac{\pi}{2}$.

$$\therefore \frac{\pi}{2} = \sqrt{\frac{W}{EI}} \cdot \frac{l}{2},$$

from which we get

$$W = \frac{\pi^2 EI}{l^2},$$

which is *Euler's formula* for *long* columns with round ends. Transposing equation (2) we get for the equation of the elastic curve

$$y = \delta \left\{ 1 - \cos\left(\sqrt{\frac{W}{EI}} \cdot x\right) \right\}.$$

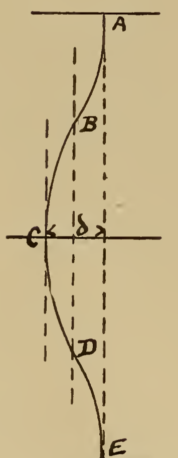

FIG. 48.

59. A column loaded with W as per the above formula, will *just* retain any deflection which may be given it, therefore this value of W is called the *critical load,* for the column will straighten out if there be any decrease in this load, and for any *increase* it will keep on bending until it breaks. This is the load then which puts the column in neutral equilibrium. The bending of columns is perfectly uniform, so we can derive from the formula of the preceding article, which is for a column with rounded or pivoted ends, the formula for columns with one or both ends *fixed.* Let Fig. 48 represent the elastic curve of a column with both ends *fixed.* There will

be two points of inflection, B and D, and as the bending is perfectly uniform these points will be at quarter the length of the column from the ends. The part BCD will represent the elastic curve of a column such as we considered in the preceding article, so the *critical load* of a column with *fixed* ends will be the same as for a column of *half its length* with pivoted ends, or,

$$W = \frac{4\pi^2 EI}{l^2}.$$

The part $BCDE$ of Fig. 44 would approximately represent the elastic curve for a column with *one* end (E) *fixed;* so the *critical load* for it would be the same as for a column of two-thirds its length with pivoted ends, or

$$W = \frac{9\pi^2 EI}{4l^2}.$$

This latter formula is not quite so accurate as the others, for the ends are not in the same vertical line.

It has been shown by experiment that when the length of a pillar is greater than 100 diameters, the theoretical values are closely approached, while with shorter lengths these values are much too large.

Columns having flat ends, if the ends are prevented from lateral movement, are considered as having *fixed* ends.

60. As *Euler's formula* is applicable only in the case of very long columns, another formula has been obtained in different ways by several different writers, which gives more accurate results for short columns. Obviously a column may be so short as to fail by crushing alone, in which case the crushing load would be $W = fA$ where f is crushing strength of the material and A the area of the cross-section. For ordinary columns, then, W must lie between fA and the value given by *Euler's formula.*

The following formula is most used for short columns. If p_1 be the compressive stress due to W, and p_2 the fiber stress due to bending, the *maximum* stress will be $p_1 + p_2$; this must not exceed the strength of the material, so for the limiting stress $f = p_1 + p_2$.

If A is the cross-sectional area of the column $p_1 = \frac{W}{A}$. The equation of bending $\frac{p}{y} = \frac{M}{I}$ gives us $p_2 = \frac{My}{I}$, remembering that y in this formula is the distance from the neutral plane to the extreme fiber of the section,

$$f = \frac{W}{A} + \frac{My}{I}.$$

From Fig. 48, if the maximum deflection be δ, we have $M = W\delta$, and as in *symmetrical* bending δ varies as $\frac{l^2}{y}$ we have $\delta = \frac{l^2}{cy}$ (the constant is always a fraction so may be put in this way); substituting these values and remembering I is equal to A multiplied by a constant squared, or, $I = Ak^2$, we have

$$f = \frac{W}{A} + \frac{Wl^2y}{cyI} = \frac{W}{A}\left(1 + \frac{l^2}{ck^2}\right),$$

from which

$$W = \frac{Af}{1 + \frac{l^2}{ck^2}}.$$

This is known as *Rankin's formula* for columns and struts with *fixed* ends.

61. We see that when l approaches zero, in the above formula, W approaches Af, and as $\frac{l}{k}$ increases, W will approach the value given by *Euler's formula*.

The formula of Art. 60 is for a column or strut with *fixed* ends; if both ends are rounded or pivoted, we must divide the value of c by 4; if *one* end is rounded c is divided by 2, so that the formulæ for different methods of end supports are

Flat ends $$W = \frac{Af}{1 + \frac{l^2}{ck^2}}.$$

One round end $$W \ldots\ldots = \frac{Af}{1 + \frac{2l^2}{ck^2}}.$$

Both ends round $$W = \frac{Af}{1 + \frac{4l^2}{ck^2}}.$$

Values of the constants c and f, for several materials are

	f	c
Hard steel	69,000 lbs. per sq. in.	20,000
Structural steel	48,000 " "	30,000
Wrought iron	36,000 " "	36,000
Cast iron	80,000 " "	6,400
Wood	7,200 " "	3,000

Examples:

1. A hollow cast-iron column, fixed at the ends, is 20 ft. high and has a mean diameter of 1 ft. It is to carry 100 tons. Factor of safety 8. What is the thickness of the metal?
Ans. 1 in.

2. What is the crushing load of a wrought-iron pillar 10 ft. high, 3 ins. in diameter, and with rounded ends?
Ans. 30 tons, nearly.

3. What is the crushing load of a cast-iron column having flat ends, being 15 ft. long and 6 ins. in diameter?
Ans. 350 tons.

4. A wooden strut, 12 ft. long, supports a load of 15 tons. If the section is square what must be the side of the square allowing a factor of safety of 10?

Ans. 9.25 ins.

5. A hollow wrought-iron column with flat ends is 20 ft. long, 10 ins. outside diameter, 7 ins. inside diameter. What load will it carry?

Ans. 616 tons.

6. A solid steel column with round ends is 6 ins. in diameter and 37 ft. long. What load will it bear?

Ans. 90,000 + lbs.

7. A square wooden column with fixed ends is 20 ft. long and carries a load of 9500 lbs., with a factor of safety of 10. What is the side of the square? (*Euler's formula.*)

Ans. 3.24 ins.

8. If in example 2 the pillar were of rectangular section of breadth, double the thickness, what sectional area would be required for equal strength?

Ans. 9.4 sq. ins.

CHAPTER XII.

STRESS ON MEMBERS OF FRAMES.

62. We have thus far considered the strength of single pieces of material; let us now investigate the methods for finding the stress on any of the parts of a loaded structure consisting of several members. This can be done graphically or by calculation, and both ways will be considered. We will understand by the word structure anything built of separate parts, called members, between which there is to be no motion. Each member must be strong enough to withstand *all* the forces to which it will be subjected without permanent deformation.

Forces are either external or internal.

External forces acting on a *structure* are (1), the weights of the members; (2), the loads carried, such as the traffic crossing a bridge, rain and wind pressure on a roof, weight lifted by a crane, etc.; and (3), the supporting forces. The *internal* forces are the resistances offered by the members to distortion.

The external forces acting on any *member* of a structure are its weight, the load and supporting forces *and* the forces exerted on it by other members.

We shall make use of the following rules for the equilibrium of a structure:

(1) The algebraic sum of *all* the external forces taken together must equal zero.

(2) All the forces, both external and internal, acting on any member must balance.

(3) All the external forces *on one side* of any imaginary

complete section of a structure must balance all the internal forces on the same side of the section.

(4) The algebraic sum of the moments about *any* axis of *all* the forces both external and internal *on one side* of any section through a structure must equal zero.

All members of a structure are connected together by *joints.*

By the word *frame* will be meant a structure which has *frictionless* joints. Frames cannot actually exist, for there is always frictional resistance; however, many structures closely approach them, and whatever error there is may be considered somewhat as a factor of safety.

A member of a structure which is in *tension* is called a *tie;* if under *compression,* a *strut.*

If two or more members of a frame are connected at a joint, the stress in the members is considered as acting at the *center* of the joint; each member necessarily having a joint at each end to keep it in equilibrium, and obviously the stress in the member will be of the same amount throughout and will act along the axis, the effect on the joint at one end being in the opposite direction to the effect on the joint at the other end.

If we consider a single joint of a structure, the resultant of the stresses in the members forming the joint must be equal in amount and opposite in direction to the load on that joint, for otherwise the joint could not be in equilibrium.

All frames will be considered as loaded *at the joints.* In the case of continuous loads, such as the weight of a member, the equivalent parallel forces will be considered as acting at the joints, thus a uniformly loaded member would be considered as having half the total load acting at each end.

63. As a simple frame we will first consider a common triangular roof truss in which the weight of the roof itself forms

the load, say w lbs. per ft.-run on each rafter. On the member AB there will be wa lbs. of which half will act at A and half at B. On the member BC there will be wb lbs., of which half will act at B and half at C; so that at B there will be a load of $\frac{w}{2}(a+b)$ lbs. Now the supporting forces P and Q will have to sustain this whole load and their value is obtained just as we found it in the case of loaded beams; the moment *arm* of the load at B being the projection of the member AB on AC, or BC on AC; the lengths being found from our knowledge of distances, angles, etc. Now the lines of

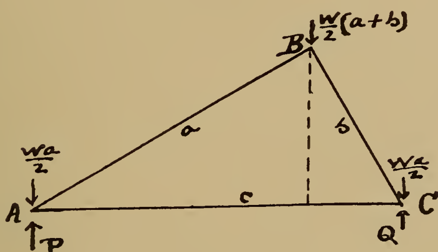

FIG. 49.

action of the load, $\frac{wa}{2}$ at A, and of the supporting force P are identical, but the directions of the forces are opposite, so we can use their difference for P without making any change in the stress of the members of the structure (this being the resultant of the two forces) and thus reducing the number of external forces to three. (It is convenient in calculating the supporting forces to ignore the load which acts on the lower end of the rafters.) We have now to find the stress in the members AB, BC, and AC, and we can do this (1) by calculating directly from our knowledge of the loads, angles and distances; (2) by means of the Ritter section, and (3) by a graphic method. Each of these methods will be explained in

the three following articles, but each method must be known thoroughly, for with complicated frames it is necessary to employ more than one of the methods to get results. Before proceeding we will introduce a system of lettering which will be convenient, and which should be used in all cases. Referring to Fig. 50, the external loads at the ends of each member have been indicated by arrowheads, between each one of these external loads we place a capital letter, and in each *space* made by the members of the frame we place a small

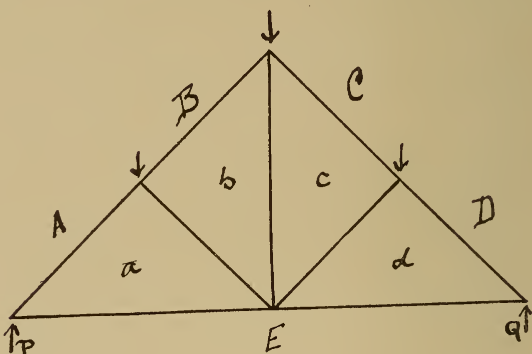

FIG. 50.

letter. To indicate any *force* we write or mention the letters on each side of it, for example, the left supporting force is EA, the right one DE, the load at the peak BC, etc. The *internal* forces will all have at least one *small* letter, though both may be small; thus the stress in the vertical member from the peak down is bc, that in the member inclined to the left from the peak is Bb, the one inclined to the right Cc, etc. With this method we letter the *forces*, which will be found most convenient when using the graphic method for finding their values.

64. As an illustration, we will find the stresses in the members of a triangular roof truss of span 25 ft., rafters of 20 and 15 ft. respectively, and which has a load of 100 lbs. per ft.-run on the rafters. The loads, additional dimensions and supporting forces as indicated in Fig. 51 are found by the methods of the preceding article,

$$\cos\theta = \sin\phi = \tfrac{16}{20} = \tfrac{4}{5},$$

$$\sin\theta = \cos\phi = \tfrac{12}{20} = \tfrac{3}{5}.$$

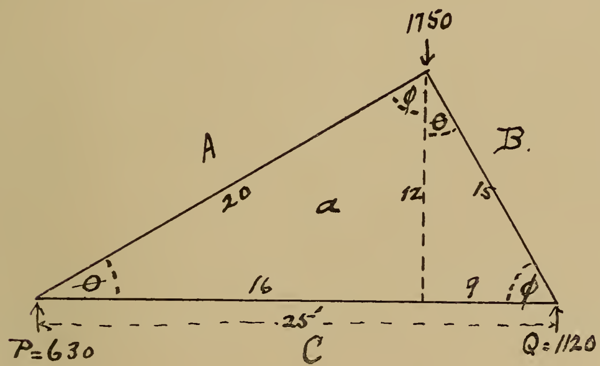

FIG. 51.

For the equilibrium of the joint at the peak the sum of the vertical components of the stresses Aa and Ba must be equal to the load 1750, but must act in the opposite direction, therefore,

$$Aa\cos\phi + Ba\cos\theta = 1750, \text{ or, } \tfrac{3}{5}Aa + \tfrac{4}{5}Ba = 1750. \quad (1)$$

Considering the joint at the right support, the sum of the horizontal and of the vertical components of the forces acting there must be zero, or as before the vertical component of Ba

must equal Q, and the horizontal component must equal the stress Ca. Resolving, horizontally,

$$Ca = Ba \cos \phi, \text{ or, } Ca = \tfrac{3}{5}Ba. \tag{2}$$

Vertically,

$$Ba \sin \phi = 1120, \text{ or, } \tfrac{4}{5}Ba = 1120. \quad \therefore Ba = 1400 \text{ lbs.}$$

Substituting in (1)

$$\tfrac{3}{5}Aa + \tfrac{4}{5} \,.\, 1400 = 1750. \quad \therefore Aa = 1050 \text{ lbs.}$$

Substituting in (2)

$$Ca = \tfrac{3}{5} \,.\, 1400 = 840. \quad \therefore Ca = 840 \text{ lbs.}$$

Now it is obvious that the stresses Aa and Ba will be compressive, and that Ca will be tensile, but in many cases it is not obvious, and with the graphic method will be shown a way of accurately distinguishing. Taking the joint at the peak, the external load acts down, so the resultant of the stresses Aa and Ba must act *up;* to do this they must *push* on the joint; a stress then which *pushes* on a joint is *compressive* and one which *pulls* is *tensile.*

This method of finding stresses will hereafter be called the method by calculation, to distinguish it from Ritter's method.

65. Ritter's Method.—This may be called the method of sections, as it involves our fourth rule for the equilibrium of a structure. The moments may be taken about *any* axis and if possible we choose an axis about which the moments of all the unknown stresses cut by the section *except one* disappear. We will solve the problem of the preceding article by this method and to do so will first consider a section HH as shown in Fig. 52. If now we take moments about the joint at the peak, the moments of the stresses Aa and Ba will be zero and we will have *to the left* of the section only the external force P and the stress Ca to deal with; or to the right the external force Q and the stress Ca, there being no moment of the

1750-lb. load about the joint at the peak. Taking moments then the *perpendicular* distance from the peak to the stress Ca being 12 ft., and that to the line of action of the force P being 16 ft., we have

$$630 \times 16 = Ca \times 12, \text{ or, } Ca = 840, \text{ as before.}$$

Taking moments about the joint at the right support will eliminate the stress Ca, and still working with the forces to the *left* of the section we have

$$630 \times 25 = Aa \times 15, \text{ or, } Aa = 1050, \text{ as before.}$$

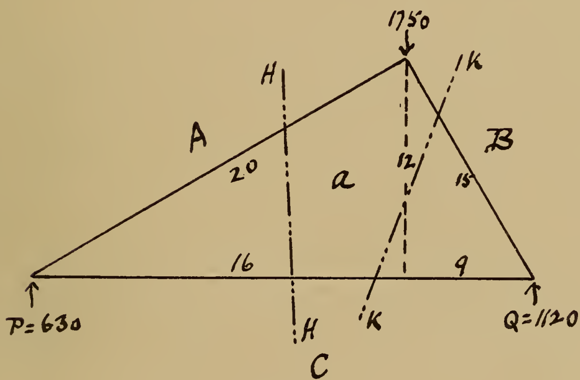

FIG. 52.

Care must be taken to use the *perpendicular* distance to the line of action of the forces. We have used 15 in this case because, it will be noticed, this frame is right-angled at the peak.

To get the stress in Ba we must use another section, for it is clear that the only internal stresses involved are those of the members through which the section passes. We will use the section KK and take the moments about an axis through, let us say, the point L. (We may use any point.) Here

the perpendicular distance to the stress Ba is 7.2 ft., and to the supporting force Q it is 9 ft.; so

$$1120 \times 9 = Ba \times 7.2, \text{ or, } Ba = 1400, \text{ as before.}$$

Instead of choosing L, for the axis of moments, we could just as well have taken the joint at the left supporting force from which the moments are

$$1120 \times 25 = Ba \times 20, \text{ or, } Ba = 1400.$$

This method will be found convenient where the stress in particular members is desired, and is frequently necessary in the graphic method.

66. The graphic method, invented by Clerk Maxwell, is based upon the principle of mechanics that a number of forces acting at a point are in equilibrium only when they can be represented in amount and direction by the consecutive sides of a *closed* polygon. In all frames the forces acting at the joints must be in equilibrium, and the line of action of the internal forces must be in the direction of the axis of the member in which they are found. All the *external* forces taken in order around the frame will form a closed polygon, because by our first rule for the equilibrium of structures they must balance. Draw carefully a diagram of the frame and at each joint indicate by an arrowhead the load, if there be one; indicate also the supporting forces. This figure is called the *frame diagram,* and will be denoted by F. D.

The diagram representing the polygons of forces acting at the joints is called the *reciprocal diagram,* and it will be denoted by R. D. Having determined the amount and direction of the external forces, plot them to a convenient scale, forming the external force polygon. In case the loads are all vertical, as in the example of Art. 63, the external force polygon is a vertical straight line, as in Fig. 53, the load 1750 lbs. being represented by AB, the supporting force Q,

acting up, by *BC*, and *P* by *CA*, all of which forces have the same distinguishing letters in the R. D. as in the F. D. Now the stress in the left-hand rafter is in the direction of the rafter itself, so from *A* of the external force polygon draw a line parallel to this rafter to represent the line of action of the stress *Aa*, and from *B* draw a line parallel to the right-hand rafter to represent the line of action of the stress *Ba*. The intersection of these two lines will be the point "a" and if we connect *C* and *a*, the line will be found to be parallel

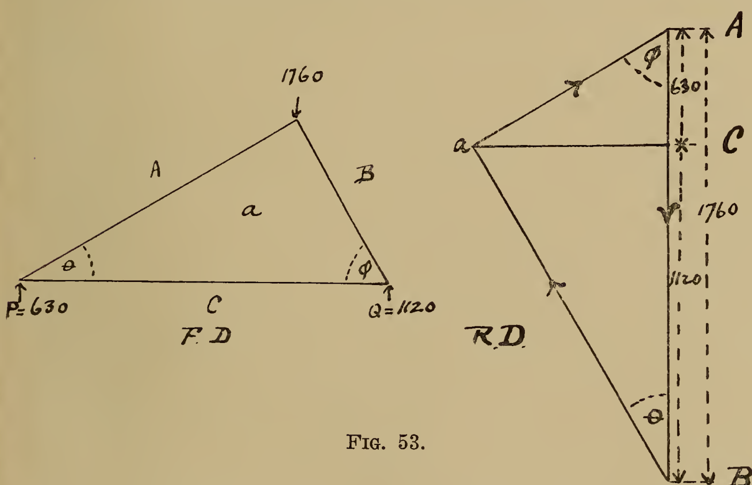

FIG. 53.

to the tie rod of the frame, and *Ca* will represent the stress in the tie rod to the same scale as that used to plot the external force polygon, just as *Aa* and *Ba* will represent the stress in the rafters. This figure is called the Reciprocal Diagram, and that the lines do represent the stresses may be proved, for the angles *BAa* and *ABa* are equal respectively to ϕ and θ of the F. D. by construction, and as *AC* is to scale equal to 630, *Aa* will equal $630 \sec\phi = 630 \times \frac{5}{3} = 1050$; *Ba* will equal $1120 \sec\theta = 1120 \times \frac{5}{4} = 1400$; and *Ca* will

equal 630 tan ϕ, or 1120 tan θ, either of which will give 840. The results are then the same as those obtained by other methods. Of course, if the diagram is drawn accurately, the stresses may be *measured* off to scale.

To find the kind of stress in any member, notice that the forces acting on the joint at the peak are the load AB, and the stresses Aa and Ba. If we pick these lines out on the R. D., they will form the polygon of forces (in this case a triangle) for the joint at the peak. Knowing the direction of *one* of these forces, we will indicate it by an arrowhead; for example, we know AB acts down, so we put an arrowhead pointing down on the line AB of the R. D. For the equilibrium of the joint at the peak the direction of the forces acting on it will be indicated if we suppose the arrowhead of AB to move in order around the sides of the polygon for this joint, starting in the direction in which it points. When on each side it will indicate the direction in which the stress for the corresponding member *acts on the joint,* and remembering that a *push* indicates compression and a pull tension, we can at once state what kind of stress exists in any member acting on the joint. The arrowheads are marked in the R. D. of Fig. 53 for the joint at the peak. They have been indicated for a single joint only, because if we consider any other joint involving the stress of any of the members acting on this one we would have two arrowheads on the same line pointing in opposite directions. This at first glance appears to be incorrect, but when we remember that the stress in any member acts in opposite directions on the joints at its two ends, and that we would now be working with a different joint, the apparent inaccuracy clears itself up; for example, if we consider the joint at the right-hand support, we will find the arrowhead for the stress Ba pointing toward B in the R. D., which we know to be correct, as the right-hand rafter is in compression and therefore *pushes* on this joint.

Examples:

1. The slope of the rafters of a simple triangular roof-truss is 30°. What is the stress in each member when loaded with 250 lbs. at the peak?

Ans. Rafters, 250 lbs.; tie rod, 216.5 lbs.

2. A beam 15 ft. long is trussed with steel tension rods and a strut at the middle forming a simple triangular truss 2 ft. deep. What is the stress on each member when loaded with 2 tons at the middle?

Ans. Strut, 2 tons; tension rods, 3.88 tons; thrust on the beam, 3.75 tons.

3. A small brow, 6 ft. broad and 20-ft. span, carries a load of 100 lbs. per sq. ft. of platform. It is supported by two simple triangular trusses 3 ft. deep. Find the stress in each member.

Ans. Strut, 3000 lbs.; tie rods, 5220; thrust on beams, 5000 lbs.

4. The rafters of a simple triangular roof-truss slope 30° and 45°; span 10 ft., the rafters are $2\frac{1}{2}$ ft. apart, and the roof weighs 20 lbs. per sq. ft. Find the stress on each member.

Ans. Stress on tie rod, 198 lbs.

5. A small brow, 4 ft. broad and 20-ft. span, carries a load of 100 lbs. per sq. ft. of platform. A load of 1 ton passes over the brow; what is the stress in the members when this load is in the middle of the brow?

Ans. Compression of strut, 3120 lbs.

6. The tie rod of a simple triangular roof-truss is 25 ft. long and inclined at 30° to the horizontal. The supports are at its ends and the rafters from its ends are 20 and 15 ft. long and are loaded with 30 lbs. per ft.-run. What is the stress in each member?

Ans. Stress in rafters, 148 + lbs. and 512 + lbs.; tie rod, 174 + lbs.

CHAPTER XIII.

FRAMED STRUCTURES—Continued.

67. A roof truss with a vertical member from peak to tie, has flooring laid on the horizontal ties, in addition to the load on the rafters, so that when divided at the joints the load will be as shown in Fig. 54.

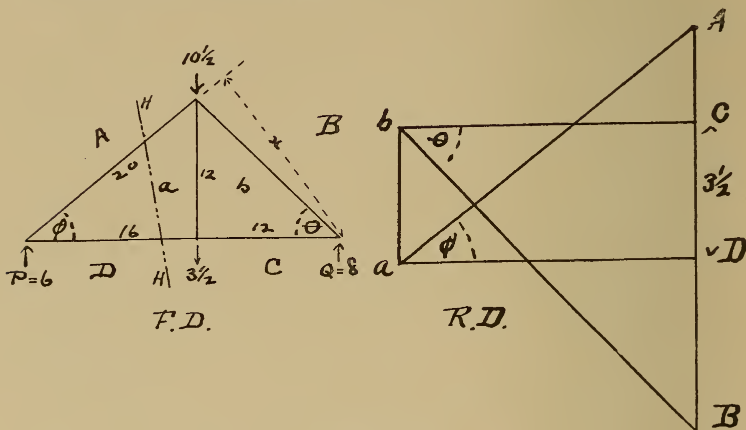

FIG. 54.

Proceeding in the usual way we get the R. D. as shown, and from it

$$Aa = 10 \text{ tons C. (compression).}$$
$$Bb = 8\sqrt{2} \text{ tons C.}$$
$$ba = 3\tfrac{1}{2} \text{ tons T. (tension).}$$
$$Da = Cb = 8 \text{ tons T.}$$

To get the stress Aa, for example, by the method of sections: taking a section HH, and moments about the joint at the

right-hand support, using forces to the left of the section, we get, x being equal to 16.8 ft.,

$$P \times 28 = Aa \times x, \text{ or, } Aa = 10 \text{ tons, as before.}$$

The method by calculation is seldom used as compared to the other methods it is complicated.

68. A "King-Post Truss," slope of rafters 45°, and having struts to the middle points of the rafters as shown in Fig. 55, has a load of 8 tons uniformly distributed on each rafter.

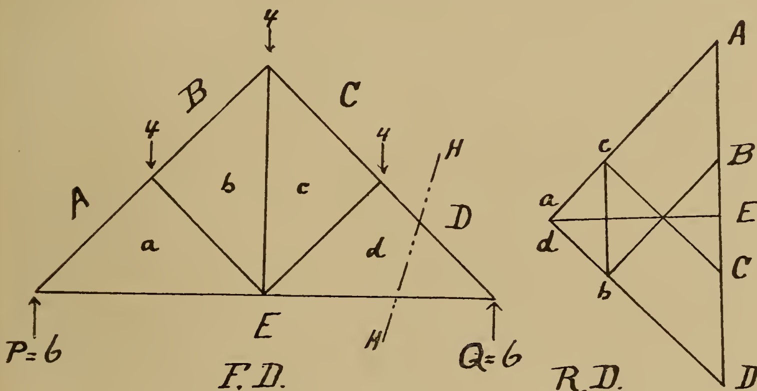

FIG. 55.

Find the stress in the members. The loads are as shown in the F. D. The R. D. gives us:

$$\begin{aligned} Aa &= Dd = 6\sqrt{2} \text{ tons C.} \\ Bb &= Cc = 4\sqrt{2} \text{ tons C.} \\ ab &= cd = 2\sqrt{2} \text{ tons C.} \\ Ea &= Ed = 6 \text{ tons T.} \\ Bc &= 4 \text{ tons T.} \end{aligned}$$

To get the stress Ed, for example, by method of sections, take section HH, moments about joint at peak, calling span S.

$$Q \times \frac{S}{2} = Ed \times \frac{S}{2}. \quad \therefore Ed = 6 \text{ tons, as above.}$$

69. In the preceding example, suppose the roof had also to sustain a horizontal wind pressure of 6 tons, uniformly distributed on the right-hand rafter. Putting in *all* the loads they would be as shown in Fig. 56, and the walls would have

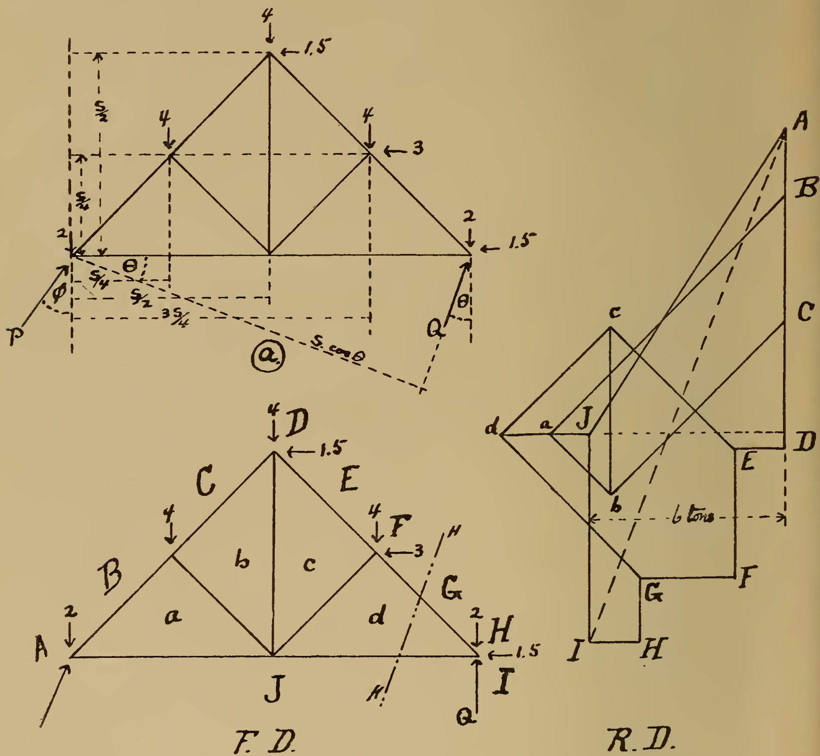

FIG. 56.

to sustain in addition to the vertical load a lateral pressure of 6 tons. The resultant of the loads is not now vertical, therefore, our supporting forces will be inclined. Obviously, the effect is greatest on the left-hand support, so the inclina-

tion of P will not be the same as that of Q. Calling the angles that P and Q make with the vertical, ϕ and θ respectively, and the span S, and taking moments about the joint at the left support (Fig. 56, *a*),

$$QS \cos \theta = \frac{S}{4} \times 4 + \frac{S}{2} \times 4 + \tfrac{3}{4}S \times 4 + 2S - \frac{S}{4} \times 3 - \frac{S}{2} \times 1.5.$$

$$\therefore\ Q \cos \theta = 6.5,$$

and, by the same method,

$$P \cos \phi = 9.5,$$

which shows the sum of the vertical components of the supporting forces to be equal to the vertical loads as it should be, also

$$P \sin \phi + Q \sin \theta = 6 \text{ tons},$$

or the horizontal components equal the horizontal loads. Proceed now with the external force polygon to and including the force HI. The next two forces are the supporting forces, we know the external forces give us a closed polygon, therefore, the resultant of the two supporting forces is the dotted line IA. We know the *sum* of the *horizontal* components of the two supporting forces is 6 tons, so we will draw a line parallel to the direction of AB, BC, etc., at a distance from it equal to 6 tons as per our scale. We also know the *vertical* component of the force P is 9.5 tons, so laying off from A, toward D the distance 9.5 per scale, if through this point we draw a horizontal line, the point J must be somewhere on it, and also somewhere on the first line. Therefore, J is located at their intersection. Connecting J and I we find the line is vertical and therefore that the supporting force at Q is vertical and equal to 6.5 tons. Connecting J and A we have the supporting force P both in magnitude and direction, and find it equal to 11.236 tons and at 32° 16′ 32″ with the vertical.

Of course practically the right-hand support would sustain some of the horizontal load, due to friction and the way the roof is secured to the walls, but if the left-hand wall is built strong enough to take *all* the horizontal effect our structure will be safe. As we *can always resolve any oblique force horizontally* and *vertically,* we may assume that the supporting wall, on the side *from which the horizontal component comes,* has a roller on its top so that the supporting force on that side will *have* to be vertical.

Proceeding now, we complete the F. D. by putting in the supporting forces, then finish the R. D. as shown in the figure and find the stresses to be as follows:

$Ba = 7.5\sqrt{2}$ tons C.
$Cb = 5.5\sqrt{2}$ tons C.
$Gd = 4.5\sqrt{2}$ tons C.
$ba = 2\sqrt{2}$ tons C.
$Ja = 1.5$ tons T.
$Ec = 4\sqrt{2}$ tons C.
$bc = 5.5$ tons T.
$cd = 3.5\sqrt{2}$ tons C.
$Jd = 3$ tons T.

To get the stress cd by the method of sections, we must know the stress Jd, then take a section KK and use the forces to the *right* of the section taking moments about the joint at the peak. For Jd, using the section HH,

$$Jd \times \frac{S}{2} = 6.5 \times \frac{S}{2} - 2 \times \frac{S}{2} - 1.5 \times \frac{S}{2},$$

or,

$$Jd = 3 \text{ tons . T.};$$

now, using section KK (a section through Ec, cd and dJ),

$$cd \times \frac{S}{2\sqrt{2}} = Jd \times \frac{S}{2} + 1.5 \times \frac{S}{2} - 6.5 \frac{S}{2} + 2 \times \frac{S}{2} + 4 \times \frac{S}{4} + 3 \times \frac{S}{4}$$

from which $cd = 3.5\sqrt{2}$. C., as before.

70. An inverted "*Queen-Truss*" is loaded, as shown in the F. D. of Fig. 57, with unequal loads over the two struts. With this loading the R. D. is as shown. The stress in the diagonal is *tension*. If the loads had been reversed, *i. e.*, $2W$ for BC, and W for AB, the stress in the diagonal would have been *compressive*. If the diagonal member had been omitted, the R. D. would not have been a closed figure (except in the case of *equal* loads over the struts), showing that the frame

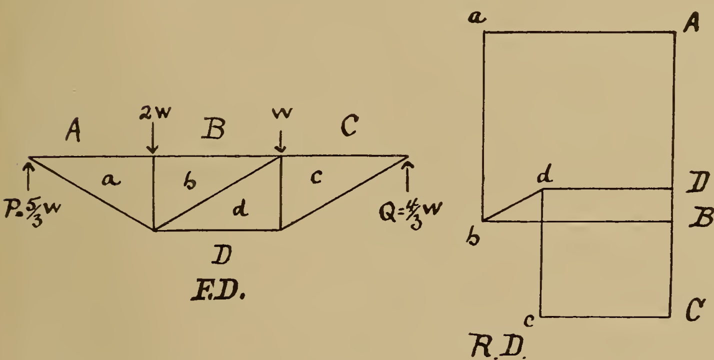

FIG. 57.

would not then be in equilibrium. The diagonal member is then necessary in bridges of this kind, because of the travelling loads they must carry.

71. A *Bollman* truss is 24 ft. long, 3 ft. deep and carries a uniform load of 3 tons per ft.-run.

The F. D. is as shown in Fig. 58. Proceeding with the R. D. we draw the external force polygon, the *direction* of the stresses *Aa, Bd, Cf, Dg, De, Dc,* and *Db*. Here we have to stop for we have none of the points represented by the small letters. The simplest way out of our difficulty will be to find

the amount of stress in one of the members by the method of sections. Taking the section *HH*, moments about the point where the two *long* tension members cross (marked with circle) and with the forces to the left of the section,

$$Bd \times \tfrac{9}{4} = 24 \times 12 - 24 \times 4, \text{ or, } Bd = 85\tfrac{1}{3} \text{ tons.}$$

Laying this value off to scale on the line *Bd* gives us the point *d*, after which we have no difficulty in obtaining the R. D. shown.

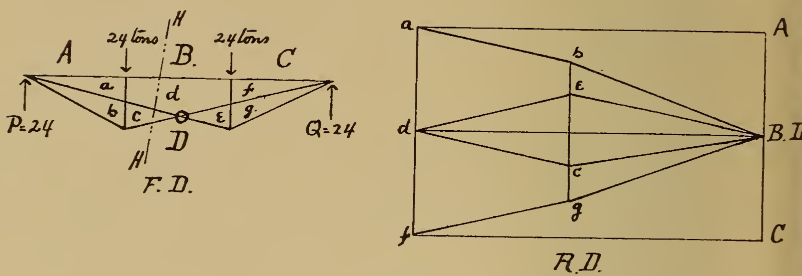

FIG. 58.

72. The force diagram of Fig. 59 shows an *N* girder of four panels with a uniform load over the lower platform. Having drawn the external force diagram the stress *Ea* being horizontal, the point *a* must lie somewhere on a horizontal line through *E;* the stress *Aa* being vertical the point *a* must lie somewhere on a vertical line through *A*. Therefore the point *a* coincides with the point *E* of the R. D., or the stress *Ea* is equal to zero with this loading. The member is necessary to the bridge, however, for even if it did not support part of the platform it would be required to keep the frame from turning about the upper joint over the left support. The stress *Bh* also proves to be equal to zero in the same way. Proceeding now we get the R. D. as shown and finding the stress *ed* is also zero. In this case, too, the member is neces-

sary, for *with a joint* at the middle of the top boom the stresses Ad and Ae being compressive would cause it to turn, and *without the joint* the necessary length of the member would cause it to buckle. Having now the R. D. the stresses in the members are easily obtained. In designing N girders we must consider the effect of *travelling* loads, for the bridge in addition to sustaining its own weight must be strong enough to sustain the stresses caused by the loads which cross it. The method of constructing bridge platforms (the planks

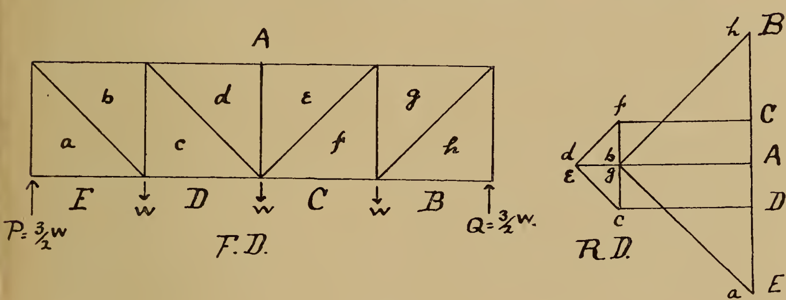

FIG. 59.

or railroad ties being placed across stringers between joints) puts practically all the bending stresses on the horizontal parts and all the shearing stresses are sustained by the diagonals (the shearing stress being equal to the vertical component of the stress in the diagonal). Therefore, considering only the weight of the structure, the diagonals at the left of the middle are inclined to the left and those to the right of the middle to the right, because those directions put them in *tension*. (For the same load a rod in tension requires less cross-sectional area than one in compression, hence less material, less weight and less cost.) If we consider a bridge of this kind with a concentrated load moving across it, referring to Fig. 41, Art. 50, the load entering the bridge at the left

end, we see that *positive* shearing force predominates at the left end and *negative* at the right, the curve in this figure being for the *travelling* load only. Therefore the *total* shearing force due to *both* the weight of the structure and the travelling load, will change sign from plus to minus at some position of the travelling load other than the middle of the bridge. Obviously, this will change the *kind* of stress in the diagonal at this point and the dimensions of the diagonal here must be made such as to sustain the new stress, *or* we must put in a second diagonal inclined in the opposite direction. The latter method is used because of the increase of weight necessary with a diagonal under compression, and also because a tension rod is less expensive and quite as efficient. These extra diagonals are called *counter braces,* and the above is the reason for counter bracing some of the middle panels of N girders.

73. A *Warren* girder, angle between members 60°, carries a uniform load on the lower platform as shown in Fig. 60.

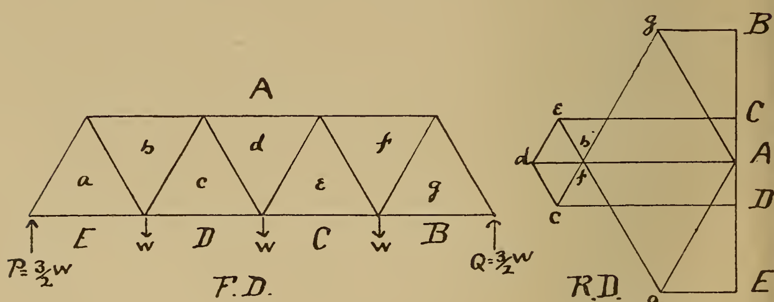

FIG. 60.

The R. D. is as shown, and the stresses in members are easily obtained from it. The remarks of the preceding article on counter bracing apply equally to this girder. If we draw

vertical lines through each joint and call the part between two vertical lines a panel, the calculation will be just the same as for the N girder.

Examples:

1. A king-post truss, slope of rafters 45°, has struts to the middle point of the rafters as in Fig. 55. $AB = 2w$; $BC = 3w$; and $CD = 4w$. Find the stress in each member by the method of sections and also from the R. D., stating the kind.

2. A bridge is constructed of a pair of Warren girders with the platform on the lower boom, which is of four divisions. 40 tons is uniformly distributed over the left half of the bridge. Find the stress in each member, stating the kind.

3. A bridge, 60-ft. span, is supported by a pair of N girders of 6 panels, height 8 ft. The platform is on the upper boom and carries a uniformly distributed load of 12 tons on the left half of the bridge. Find stress in each member, stating the kind.

4. A roof is constructed of two rafters at right angles, a horizontal member connects the *middle* points of the two rafters, the ends of this member are connected with lower ends of the opposite rafter. The joints of the peak and the middle of each rafter carry a load of 1 ton. Find the stress in each member and the kind, if the span is $20\sqrt{2}$ ft.

5. A bridge of 60-ft. span is constructed of a pair of Warren girders, the upper boom, which supports the platform, is of 6 divisions, the lower boom of 5. The platform is also supported by struts from the joints of the lower boom, and carries a load of $\frac{4}{5}$ ton per ft.-run. The supporting forces are at the ends of the upper boom. Find the amount and kind of stress in each member.

6. A Bollman truss of 48-ft. span and 12 ft. deep carries a uniform load of $\frac{3}{4}$ ton per ft.-run over the left two-thirds of the horizontal boom. Find the amount and kind of stress in each member by the method of sections and also from the R. D.

7. A bridge, 96-ft. span, is supported by a pair of *N* girders with 8 panels 9 ft. deep. The platform rests on the upper boom and is loaded with 96 tons uniformly distributed. Find the amount and kind of stress in each member.

8. A concentrated load of 30 tons is to pass over the bridge of example 7. What panels should be counter braced?

9. A Warren girder, 9 ft. long, projects from a wall. The top boom is of three divisions, the lower of two, but has a horizontal strut from the inner joint to the wall. With a load of 2 tons at the outer end of the upper boom find the stress in all the members, stating the kind.

10. Suppose the load of example 9 at each of the other two joints of the upper boom, and thence deduce the results for a distributed load of $\frac{2}{3}$ tons per ft.-run.

CHAPTER XIV.

FRAMED STRUCTURES—Continued.

74. An example of a scissors-beam truss is shown in Fig. 61. With vertical loads the R. D. shown is readily obtained.

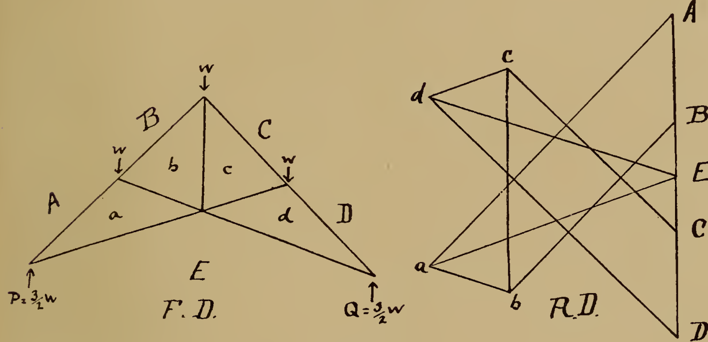

FIG. 61.

With all roof-trusses the loads due to wind pressures must be taken into account. We have seen that this may be done by resolving the wind loads into their horizontal and vertical components (Art. 69) and considering the total effect as sustained by *one* of the walls. It may also be done by combining the loads on each joint and using the resultant load on each joint in drawing the external force diagram, or by finding the R. D. for the vertical and inclined loads separately and algebraically, adding the stress due to each for the total stress in the members.

75. When the external loads and the members of a structure are all in the same plane we can *always* find the stress in

the members by means of the reciprocal diagram, for example, the F. D. of Fig. 62, shows a common crane carrying a weight W. The external forces are as shown, the forces BC and DA being obviously necessary to prevent overturning, their value being readily found by taking moments. The force CD is

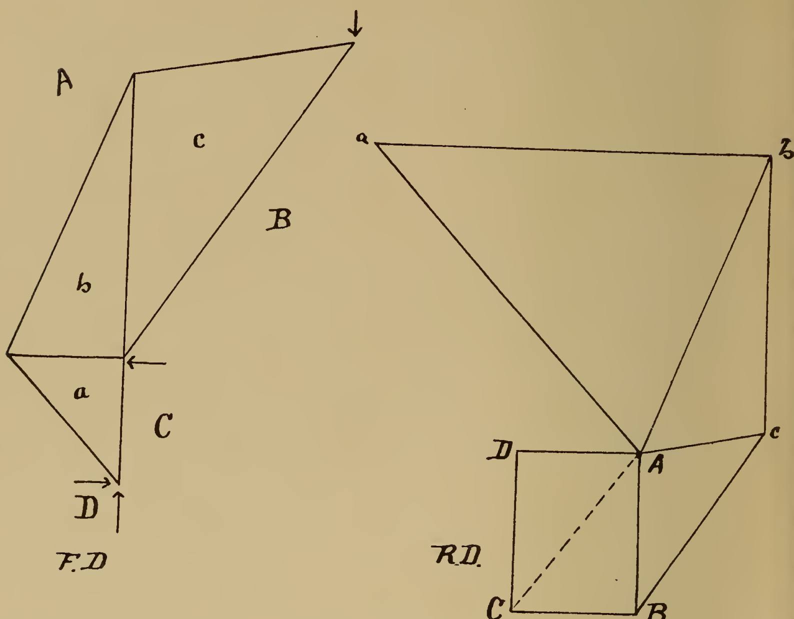

FIG. 62.

equal to the weight. The external force diagram is a rectangle. The forces CD and DA can be combined if desired, their resultant being equal to the dotted line CA. The R. D. is readily completed as shown, and from it the stresses are easily found. Cranes are usually arranged so that the load is lifted by means of a tackle, the hauling part of which leads down to a drum secured somewhere along the crane post.

The stress in this rope (which is equal to $\frac{W}{n}$, where n is the number of parts of rope between the two blocks of the tackle) augments the compression in the jib and reduces or increases the tension of the jib stay, depending on whether the drum is above or below the point where the jib joins the crane post.

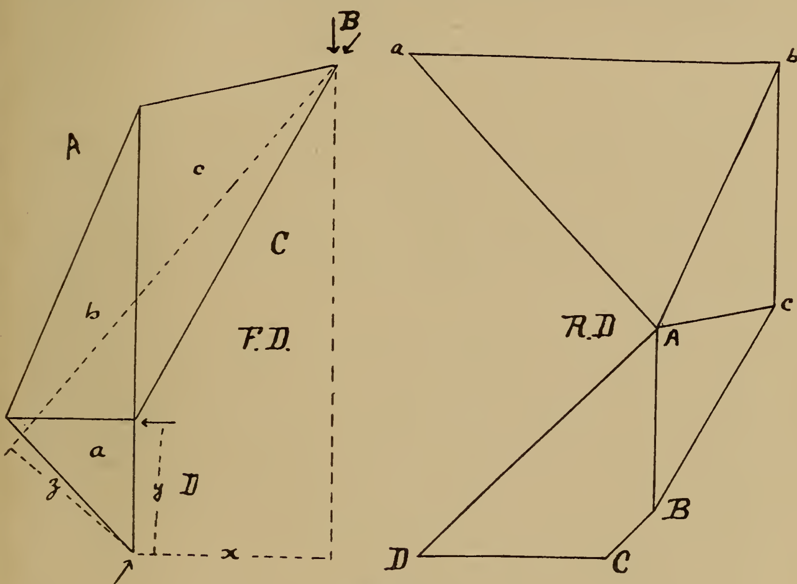

FIG. 63.

These additional loads on the members must be taken into account. This can be done by introducing an additional external load at the end of the jib, acting in the direction of the hauling part of the tackle and equal to $\frac{W}{n}$. This additional load will cause the supporting force DA of Fig. 63 to be somewhat inclined.

To draw *any* external force diagram we must know all but

one of the external forces. In this case we know only the weight lifted and the tension of the rope. We can, however, find the force CD by taking moments about the lower end of the crane post, as

$$W \times x = \frac{W}{n} \times z + CD \times y,$$

whence

$$CD = \frac{W(nx - z)}{ny}.$$

The force DA is the resultant of a horizontal force similar to DA of Fig. 62 and the slightly inclined supporting force. Except the resultant now we know all the external forces, so our external force polygon is $ABCD$, the line DA representing the amount and direction of the supporting force. The R. D. is now readily found to be as shown, and the stresses are easily obtained.

76. Incomplete Frames.—A frame having just enough members to enable it to retain its shape under any kind of loading is complete. Frames may have more members than necessary, under which circumstances some of the members are *redundant*, but frames without a sufficient number of members to enable them to retain their shape under all circumstances, with any kind of loading, are said to be incomplete. When frames are incomplete we will find the reciprocal diagram for different loadings giving intersections for the same point in two or more places; as, for example, we know the mansard roof-truss, shown in the F. D. of Fig. 64, is incomplete because there is nothing to prevent the three upper joints from turning. Suppose $W_1 = W_3$ and drawing the R. D., we find the point "a" falls in two different places on the horizontal line from E; this is clearly impossible, for the stress Ea cannot have two values. The frame then is not in equilibrium. If W_1 had not been equal to W_3 the two a's

would not have been on the same horizontal line. We *could* arrange the *loads* so that this frame would appear from the R. D. to be complete; as, assume either of the points "a" to be correct, say the right one, and from it draw lines (dotted in figure) parallel to the stresses *Ba* and *Ca*. These lines will intersect the external force diagram at points indicating loads which would put the frame in equilibrium, but the first puff

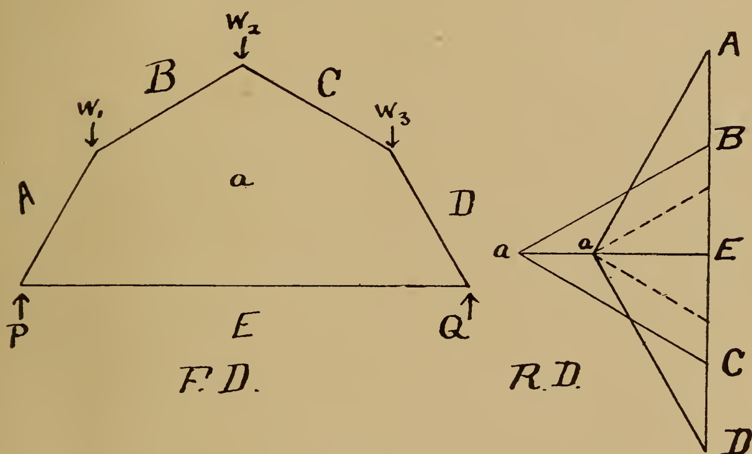

FIG. 64.

of wind would change the loading and cause the frame to collapse.

To make this frame complete, at least two additional members are necessary, for there are three joints which are likely to turn.

The two additional members, shown in the F. D. of Fig. 65, will complete the frame, as is shown by the R. D. being a closed figure and possessing no duplicate points, no matter what loads are applied.

Neither of these additional members *alone* will do this, as

will be made clear if we discard one of them, vary the loads and draw the R. D.'s.

A frame, then, to be complete, must remain in equilibrium *under any loads* which do not stress its members beyond the elastic limit.

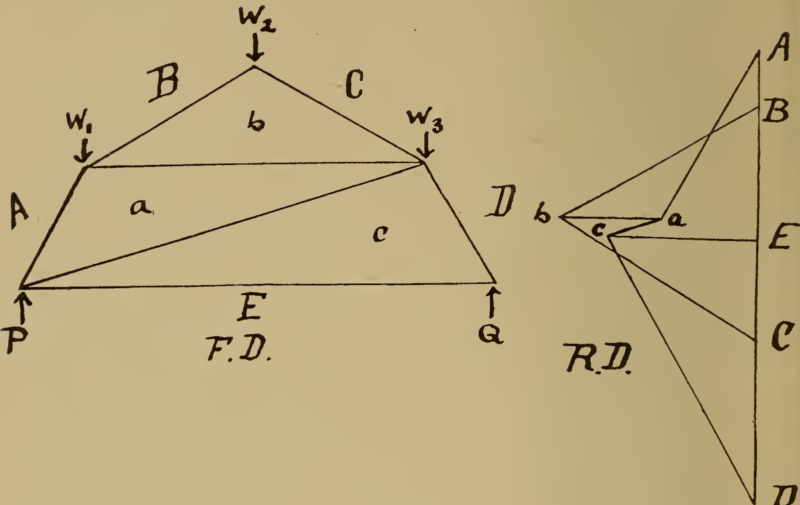

FIG. 65.

77. We have been considering in all these structures the stress which acts along the axis of the member. When members are not straight, or when they carry loads at points other than the joints, the stress on them is not simply a thrust or pull in the direction of the axis, but includes a bending and a shearing action. Fig. 66 shows a single member of a structure with two intermediate vertical loads. If we resolve all these forces along and perpendicular to the member, the components perpendicular to the member will cause bending and shearing stresses in it, and using these components, $W \cos \theta$

and $W_1 \cos\theta$ as the loads, and $P\cos\theta$ and $Q\cos\theta$ as the supporting forces, we find the shearing and bending action just as we did in Chapters VI and VII.

These intermediate loads cause also a stress along the axis of the member and this stress (the thrust T shown) between W and W_1 will be equal to $W_1 \sin\theta$ and between P and W it will be the sum of $W_1 \sin\theta$ and $W\sin\theta$, or $(W + W_1)\sin\theta$, which value is the reaction *of this member* on the one next below it.

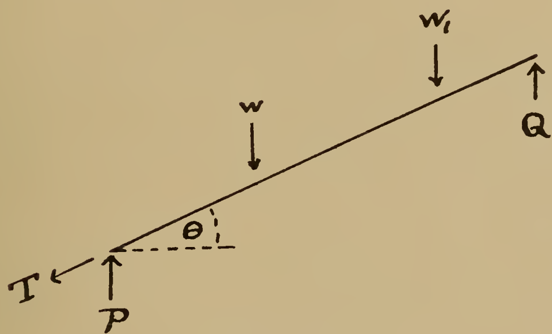

Fig. 66.

Obviously $(P + Q)\sin\theta$ causes an equal reaction on the member next above it.

Just here we can show that our assumption that the loads act on the joints is correct, for the action on these joints of the structure, considered as a whole, caused by this member is equal to a force P acting *downward* at the left end, and a force Q acting *downward* at the right end; and obviously without error we can use these forces P and Q (reversed) instead of considering the separate loads. When a member carries a uniform load, as is usually the case, the values of P and Q are each half the load and the thrust T is the value of the axial stress at the middle of the member. We have not

considered the bending and shearing stresses in members of a structure until now, because they are comparatively small, a compression member being necessarily large to sustain the thrust due to all the other members which affect it, will be little inconvenienced by the small stress due to the bending and shearing caused by its own load, and the tension members, in properly built structures, will have only their own weight to cause bending and shearing.

Examples:

1. A distributed load of 4 tons is carried by a scissors truss whose rafters, 12 ft. long, are at an angle of 90°. Two of the three other members join the lower end of our rafter with the middle of the other. The third is horizontal and joins the middle points of the rafters. Find the stress in all the members.

2. In example 4, Chapter XII, find the shearing force and bending moment and thrust for each point of each rafter and draw curves showing results.

3. A simple triangular frame, sides 3, 4 and 5 ft. long, has the 5-ft. side horizontal. Its numbers weigh 10 lbs. per ft.-run, and the inclined sides each have 50 lbs. at the middle. Draw curves of thrust, shearing force and bending moment for each member.

4. A suspension bridge carries a platform 8 ft. wide, span 63 ft. suspended by 6 equidistant tension rods. The lowest joint is 7 ft. below the highest, and the cable is formed of 6 straight members. It carries a load of 1 cwt. per sq. ft. of platform. Find the sectional areas of the cable members, allowing a stress of 4 tons per sq. in. for the material. Is this frame complete?

Ans. From left end, 3.72, 3.6, 3.5 sq. in.; middle, 3.47 sq. in.

5. A mansard roof-truss, as in Fig. 65, carries 1, 2 and 3 tons on the left, top and right joints. The rafters make angles of 45° and 30° with the horizontal. Find the stress in all the members.

6. A crane, as in Fig. 62, has the tension members *Ac* horizontal and 6 ft. long. The members *bc, ba* and *aC* are respectively 8, 6 and 4.5 ft. long. Find the stress in each member when supporting a weight of 12 tons.

MISCELLANEOUS PROBLEMS.

78. Reinforced Concrete Beams.—Concrete is much used in building at present and though its tensile strength is very low compared to other materials, the ease with which it is handled, transported and shaped, together with the fact that when reinforced with steel its tensile strength is satisfactory and the structure is non-combustible has made it most acceptable to builders. The determination of the constants for concrete is rather difficult considering the changes they suffer

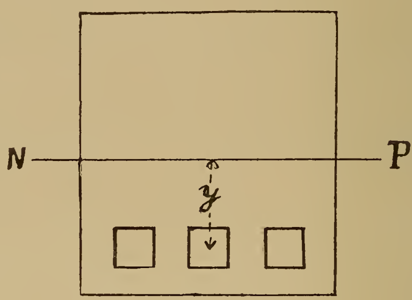

FIG. 67.

due to the different kinds of cement used and the different methods of mixing. Strength tests for the cement itself vary from 40 to 1000 lbs. per sq. in. for tension and from 1100 to 12,000 lbs. per sq. in for compression.

In all building the endeavor has been to subject the concrete to little or no tensile stress, but to have all that stress sustained by the steel reinforcement, allowing the concrete to sustain as much as possible of the compressive stress. Tests have shown the compressive strength of concrete to

vary from 750 to 5360 lbs. per sq. in. and the modulus of elasticity to range from 500,000 to 4,167,000 in in.-lb. units.

A method of finding the moment of inertia of the section of a reinforced concrete beam is as follows: Assuming that the elongation of the steel reinforcement is the same as for the concrete at equal distances from the neutral plane which will be true if the concrete adheres closely to the metal reinforcement, we have by Hook's law,

$$p_{steel} = E_{steel} \times \text{extension}$$

and

$$p_{concrete} = E_{concrete} \times \text{extension},$$

or,

$$\frac{p_s}{E_s} = \frac{p_c}{E_c},$$

from which

$$p_s = \frac{E_s}{E_c} p_c .$$

Fig. 67 represents the section of a concrete beam reinforced by three steel rods on the tension side of the neutral plane, N. P. If now we suppose the beam to be entirely of concrete and to retain the same strength and the same depth we will have to broaden the section in wake of the steel rods to conform with the fiber stress which that part of the section can now withstand. If y is the distance from the neutral axis to an elementary area, dA, of the steel, the moment of the stress about the neutral axis will be

$$y p_s dA = y p_c \frac{E_s}{E_c} dA.$$

So, if the section is to be considered homogeneous (all concrete), we must have the areas vary as the modulii of elasticity or the area of concrete which is equivalent to the present area of steel will be as E_s is to E_c. The equivalent concrete

section will therefore be something like that shown in Fig. 68. The general equation of bending gives for the fiber stress

$$p = \frac{My}{I},$$

I being the moment of inertia of this equivalent section. For example, suppose the beam of Fig. 67 to be 6 in. wide, 10 in. deep and the reinforcing rods to be $\frac{1}{2}$ in. square with their lower faces 1 in. from the surface of the concrete. Let E_s be to E_c as 15 to 1. Here the area of the steel section is $\frac{3}{4}$ sq. in., hence the equivalent area of concrete is

$$15 \times \tfrac{3}{4} = \tfrac{45}{4} \text{ sq. in.,}$$

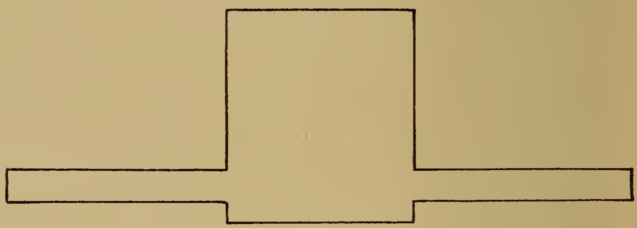

FIG. 68.

and as the rods are half an in. deep the additional breadth at this part of the section due to the steel will be $\frac{45}{2}$ ins. But there is at this part of the section beside the steel $\frac{9}{2}$ ins. of concrete, so the *whole* width of the section at the point will be

$$\frac{45}{2} + \frac{9}{2} = \frac{54}{2} \text{ ins.,}$$

and the depth of the broadened part will be the same as the steel, or $\frac{1}{2}$ in. We have now all the dimensions of the equivalent concrete section from which we find that the neutral axis is close to 4.4 ins. from the bottom of the beam, and the moment of inertia of the section about this axis is about 626 in in.-units.

It has been shown by experiment that small cracks appear on the tension side of a reinforced concrete beam as soon as the fiber stress reaches the limiting tensile strength of the concrete. From this fact it is clear that practically all of the tensile stress is supported by the steel reinforcement. With this assumption then, referring to Fig. 69, the sum of all the tensile stresses on one side of a section must equal the sum of all the compressive stresses on the same side of the section. If then A is the known area of the steel section

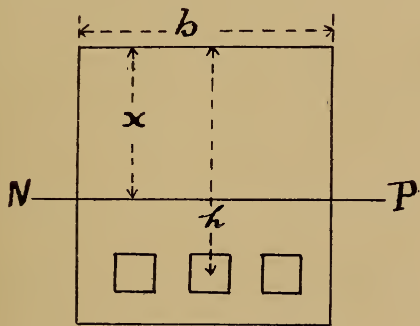

FIG. 69.

and P_s the stress in it, and bx the area of the concrete section on the compressive side of the neutral plane and p_c the maximum compressive stress in it, we have, assuming the compressive stress to vary directly as the distance from the neutral plane

$$Ap_s = \tfrac{1}{2}p_c bx. \qquad (1)$$

We know the position of the steel reinforcement, and calling its distance from the *top* of the beam h, we have the sum of the moments of the stresses about the neutral axis equal to the bending moment, or

$$Ap_s(h - x) + \tfrac{1}{2}p_c bx \,.\, \tfrac{2}{3}x = M. \qquad (2)$$

Now as the bending is considered uniform the extension of the steel on the tension side must equal the contraction of that part of the concrete on the compression side which is at the same distance from the neutral axis as the steel, therefore, by Hook's law (strains vary as distance from neutral plane)

$$\frac{p_s}{E_s(h-x)} = \frac{p_c}{E_c x}. \tag{3}$$

We now have three equations from which we can find the three unknown quantities p_s, p_c and x; p_s and p_c being the *total* stresses. If we have a reinforced concrete column under a load, the parts of the load borne by the concrete and steel can be found because the change of length of the two materials *must* be the same. We have, therefore, the equations

$$\frac{lp_s}{A_s E_s} = \frac{lp_c}{A_c E_c}, \tag{1}$$

and the load

$$L = p_s + p_c, \tag{2}$$

where l is the length of the column, A_s and A_c the sectional areas of steel and concrete respectively. Concrete columns are usually made so short that bending does not enter into the calculations, which is the case when the column is not longer than twelve times its least diameter. A difficulty in investigating the bending of reinforced concrete beams lies in the fact that the modulus of elasticity of concrete under compression decreases somewhat as the loads increase. This will evidently cause an upward movement of the neutral plane. Several theories with different assumptions have been advanced but the investigation of these beams may be considered to be still in an experimental stage. The use of the equivalent section gives fair results. Another theory assumes the stress in the concrete to vary as the ordinates of a parabola and that the modulii of elasticity for the different stresses are those found by tests for varying pressures.

79. Poisson's Ratio.—Experiment shows us that when a piece of material is stretched the area of a transverse section is reduced somewhat in addition to the change made in the length of the piece. This change of cross-sectional area is quite marked if we stretch a piece of rubber because the elongation is great, but with materials such as are used in building the change of cross-section is scarcely perceptible for the elongation is very slight. Tests show that the *lateral* expansion or contraction due to compression or tension bears a constant ratio to the change of length. This is known as Poisson's ratio. If then the extension per unit length of a round rod under tension is e, the length of its diameter d will be reduced an amount equal to ked where k is the value of Poisson's ratio. Poisson's ratio is for

Steel	.297
Iron	.277
Copper	.340
Brass	.357

If a rod of length l, and of unit square section is under a tensile stress F along its axis and a compressive stress R, perpendicular to the axis on two opposite faces, the *total* extension in the direction of the axis is that due to the tensile stress *plus* that due to the compression stress, or,

$$\text{total extension} = \frac{Fl}{E} + \frac{kRl}{E}.$$

80. Stress in Guns.—We found in Art. 10 that the hoop stress of a thin cylinder under the internal pressure of s lbs. per unit area was $p = \frac{sr}{t}$ where t was the thickness of the metal and r the radius of the cylinder. In this case the metal was considered so thin that the stress might be taken as uniform throughout the section. If, however, the metal is *thick* we cannot use this formula for the stress will not be

uniform throughout the section but will vary in some way with the distance from the axis of the cylinder.

Let us consider a closed cylinder of internal radius r_2 and external radius r_1 under an internal pressure S and an external pressure S_1 per unit area. The cylinder having closed ends there will obviously be a longitudinal stress which is readily seen to be the total pressure on an end divided by the sectional area of the cylinder. We will call this longi-

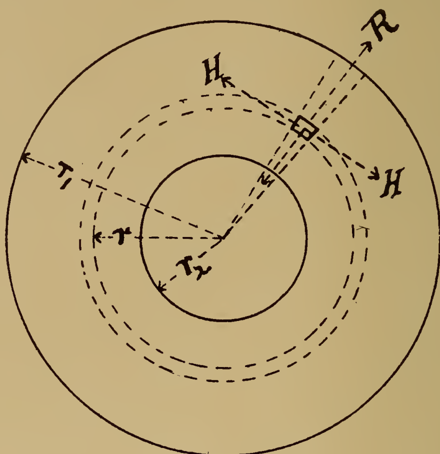

FIG. 70.

tudinal stress T and will assume it is uniform throughout the section. If now we consider the forces acting on any element of volume within the material of the cylinder the element will be in equilibrium under the action of the longitudinal stress T, the hoop stress H and the radial stress R. From the preceding article we have for a value of the total extension at the element in the direction of the axis of the cylinder

$$e = \frac{1}{E}(T - kH - kR),$$

k being the value of Poisson's ratio for the material. Now, if sections remain plane while stressed, we may assume the total extension due to these three forces constant, and T being constant, we must have H plus R equal to a constant,

$$H + R = C. \tag{1}$$

Let Fig. 70 represent a right section of the cylinder of unit thickness (perpendicular to the paper) and let the circular element shown be of width dr. If we call the radial stress at the inner surface of the element R, the hoop stress at the inner surface of the element will be (Art. 10) equal to Rr. The radial stress at the outer surface of the element will be $R + dR$ and as the radius here is $r + dr$ the hoop stress at the outer surface of the element will be $(R + dR)(r + dr)$. The hoop stress over the sectional area of the element will be $(R + dR)(r + dr) - Rr$. But we have called the hoop stress at any point H, therefore, the hoop stress on this element will be H multiplied by the sectional area of the element, hence we will have, the element being of unit thickness,

$$(R + dR)(r + dr) - Rr = Hdr;$$

or, neglecting the higher order of infinitesimals

$$rdR + Rdr = Hdr. \tag{2}$$

From (1) we have $H = C - R$ and, substituting,

$$rdR + Rdr = Cdr - Rdr,$$

multiplying both sides of the equation by r, we have

$$r^2dR + 2rRdr = Crdr,$$

or,

$$d(r^2R) = Crdr,$$

and, integrating,

$$r^2R = C\frac{r^2}{2} + C_1,$$

where C_1 is the constant of integration.

From the last equation we get

$$R = \frac{C}{2} + \frac{C_1}{r^2}, \tag{3}$$

and, substituting this value of R in (1), we get

$$H = \frac{C}{2} - \frac{C_1}{r^2}. \tag{4}$$

Now the cylinder has an internal pressure of S lbs. per unit area, and an external pressure of S_1 lbs. per unit area, so if $r = r_2$ we have R at the inside surface equal to S and if $r = r_1$ we have R at the outside surface equal to S_1 and, substituting, we get

$$S = \frac{C}{2} + \frac{C_1}{r_2^2}, \tag{5}$$

and

$$S_1 = \frac{C}{2} + \frac{C_1}{r_1^2}; \tag{6}$$

solving (5) and (6) we have

$$C_1 = \frac{r_1^2 r_2^2 (S_1 - S)}{r_2^2 - r_1^2}, \tag{7}$$

and

$$C = \frac{2(r_1^2 S_1 - r_2^2 S)}{r_1^2 - r_2^2}, \tag{8}$$

substituting the values from (7) and (8) in (3) and (4) we get

$$R = \frac{r_1^2 S_1 - r_2^2 S}{r_1^2 - r_2^2} - \frac{r_1^2 r_2^2 (S_1 - S)}{r^2 (r_1^2 - r_2^2)}, \tag{9}$$

and

$$H = \frac{r_1^2 S_1 - r_2^2 S}{r_1^2 - r_2^2} + \frac{r_1^2 r_2^2 (S_1 - S)}{r^2 (r_1^2 - r_2^2)}. \tag{10}$$

If there is no external pressure as in the cases of cast guns and hydraulic pipes, S_1 will be equal to zero and our formulas become

$$R = \frac{r_2^2 S}{r_1^2 - r_2^2}\left(\frac{r_1^2}{r} - 1\right), \tag{11}$$

and

$$H = -\frac{r_2^2 S}{r_1^2 - r_2^2}\left(\frac{r_1^2}{r} + 1\right). \tag{12}$$

From equations (11) and (12) we see that the greatest stress is at the inner surface of the cylinder and here the total extension found by the method of Art. 79 must not exceed the elastic limit of the material. It will be noticed that the values of H and R found above do not take into consideration the lateral contraction of the material mentioned in the preceding article. If we consider this lateral contraction and call H_1 the stress which would produce an extension equal to the total elongation found by the methods of Art. 79 we would have

$$H_1 = H - kR - kT.$$

For the material used in the construction of the modern naval gun the value of Poisson's ratio is $\frac{1}{3}$. Putting in this value for k and the values of H and R from equations (9) and (10), the value of T being found directly from the external and internal end pressures and the sectional area of the cylinder to be

$$T = \frac{Sr_2^2 - S_1 r_1^2}{r_1^2 - r_2^2},$$

we have, after reducing, for the *true* hoop stress

$$H_1 = \frac{r_1^2 S_1 - r_2^2 S}{r_1^2 - r_2^2} + \frac{4r_1^2 r_2^2 (S_1 - S)}{3r^2 (r_1^2 - r_2^2)}, \qquad (13)$$

which is the formula used for the hoop stress for naval guns.

81. Built-up Guns.—The guns at present used in the navy are known as "built-up" guns; that is, they are built of several pieces which are made separately. The inner piece is called the *tube*, outside of it is the *jacket*, and outside of the jacket are the *hoops*. The exterior diameter of the tube is made a little greater than the interior diameter of the jacket, the exterior diameter of the jacket a little greater than the interior diameter of the first hoop, etc. The process known as "*assembling*" consists in heating the jacket until

it expands sufficiently to just slip over the tube, after which it is allowed to cool. In cooling the jacket contracts and grips the tube firmly, putting considerable external pressure on it. The hoops are then put over the jacket in the same way. This way of securing the several parts together is called the *method of hoop shrinkage,* and obviously if two parts are assembled by this method the inner one will be under hoop compression and the outer one under hoop tension. Considering a cylinder formed of two parts, let the radii *after* assembling be r_3, r_2 and r_1, the least radius being r_3. Let R_3, R_2 and R_1 be the radial stresses at the inner, intermediate and outer surfaces. We will call the difference between the inside radius of the jacket and the outside radius of the tube *before assembling* e, then if *after* assembling e_2 is the decrease in the length of the radius of the tube and e_1 the increase of the length of radius of the jacket we have

$$e = e_2 + e_1 . \tag{1}$$

If H_2 is the hoop compression on the outside surface of the tube and H_1 the hoop tension on the inside surface of the jacket, e_2 the change of length of the radius of the tube due to the stress H_2 will be $\frac{H_2 r_2}{E}$ (Chapter I) and e_1 the change of length of the radius of the jacket due to the stress H_1 will be $\frac{H_1 r_2}{E}$, hence

$$e = \frac{H_2 r_2}{E} + \frac{H_1 r_2}{E},$$

or,

$$H_2 + H_1 = \frac{Ee}{r_2}. \tag{2}$$

There being no internal pressure in this cylinder, equation (13) of the preceding article will give us a formula for the hoop compression at the outer surface of the tube if we let

$S = 0$, $S_1 = R_2$ and $r = r_2$ at the same time changing r_2 of equation (13) to r_3 and r_1 to r_2. This gives us the equation

$$H_2 = \frac{r_2^2 R_2}{r_2^2 - r_3^2} + \frac{4r_2^2 r_3^2 R_2}{3r_2^2(r_2^2 - r_3^2)}. \tag{3}$$

There being no external pressure on the jacket equation (13) of the preceding article will also give us a formula for the hoop tension at the inner surface of the jacket if we put $S_1 = 0$, $S = R_2$ and $r = r_2$. This gives us the equation

$$H_1 = -\frac{r_2^2 R_2}{r_1^2 - r_2^2} - \frac{4r_1^2 r_2^2 R_2}{3r_2^2(r_1^2 - r_2^2)}. \tag{4}$$

Taking the factor R_2 from the right side of equations (3) and (4) all the quantities that remain are known (they being the different radii). Calling the value of these remaining quantities in equation (3) C_1 and those for equation (4) C_2 we have

$$H_2 = R_2 C_1, \tag{5}$$

and

$$H_1 = -R_2 C_2, \tag{6}$$

dividing (5) by (6) to eliminate R_2 we have

$$\frac{H_2}{H_1} = -\frac{C_1}{C_2}. \tag{7}$$

Solving the simultaneous equations (2) and (7) for H_1 and H_2 gives

$$H_1 = \frac{C_2 Ee}{(C_2 - C_1) r_2}, \tag{8}$$

and

$$H_2 = \frac{C_1 Ee}{(C_1 - C_2) r_2}. \tag{9}$$

Equations (8) and (9) give the hoop stresses at the joint and having these we can from either of the equations (3) or (4) find the value of R_2. Having R_2 equation (13) of the preceding article will give us the value of the hoop compression at the inner surface of the tube at which place the

hoop compressive stress is obviously greatest when the gun is at rest. At the instant of firing a gun the internal pressure becomes very great and both the tube and jacket will then be under hoop tensile stress, but obviously the first effect of the internal pressure must be to overcome the initial compressive stress in the tube, though it increases the tensile stress in the outer hoops where the *effect* of the internal pressure is least; thus the built-up gun can sustain greater internal pressures than the solid gun.

82. Stress Due to a Centrifugal Force.—If a body of mass m, at the end of a cord, is whirled round in a circle the cord will be put in tension, the centrifugal force as we have learned from mechanics being equal to $\frac{mv^2}{r}$ where v is the linear velocity and r the distance of the center of gravity of the mass from the axis of rotation. If n is the number of revolutions per second, the angular velocity ω will be equal to $2\pi n$. As the linear velocity is $r\omega$, the tension of the cord in terms of the angular velocity will be equal to $\frac{4W\pi^2n^2r}{g}$ $\left(m \text{ being equal to } \frac{W}{g}\right)$. Consider a fly-wheel, the spokes of which represent a comparatively small part of the total weight, so that we may consider all the weight as being at the rim of the wheel. Let the thickness of the rim be t, and its distance from the axis of rotation be r. Let the weight of the material be w lbs. per unit volume, and the breadth of the rim a units. Then the total weight of the rim will be $w \,.\, 2\pi rta$, the total radial force from the above formula is

$$\frac{w \,.\, 2\pi rta \times 4\pi^2n^2r}{g},$$

and the radial force per unit area will be

$$R = \frac{w \,.\, 2\pi rta \times 4\pi^2n^2r}{g(2\pi ra)} = \frac{w \,.\, 4\pi^2n^2rt}{g}.$$

This will give us the radial stress for a rim so thin that the stress may be considered uniform throughout the section. If, however, we consider a thick rim whose inside radius is r_2 and whose outside radius is r_1 the increment of radial stress on a circular element whose thickness is dr will be given by the above formula. We will have then for the increment of radial stress on this elemental volume

$$dR = -\frac{w \cdot 4\pi^2 n^2 r dr}{g}. \qquad (1)$$

The value of this increment having the negative sign because the radial stress decreases as the distance from the axis of rotation increases. To prove this latter statement suppose a bar of uniform section and length l rotates about an axis through its end. Let a be the sectional area, w the weight per unit volume, and x the distance of any point P from the axis of rotation. The stress at any point P is due to the centrifugal force of the part of the rod beyond P, therefore, as the center of gravity of this part is at a distance $\frac{l-x}{2}$ from the axis and its weight is $a(l-x)w$, the centrifugal force by the formula at the beginning of this article is

$$\text{C. F.} = \frac{wa(l-x)^2\omega^2}{2g}.$$

It is clear from this equation that when $x = l$ the C. F. is zero and when $x = 0$ it is a maximum. Proceeding then the integration of (1) gives

$$R = -\frac{4\pi^2 n^2 w r^2}{2g} + C_1,$$

C_1 being the constant of integration. We know the value of the radial stress at the outer edge of the rim is zero, therefore when $r = r_1$

$$0 = -\frac{4\pi^2 n^2 w r_1^{\ 2}}{2g} + C_1,$$

from which

$$C_1 = \frac{4\pi^2 n^2 w r_1^2}{2g};$$

hence,

$$R = \frac{4\pi^2 n^2 w}{2g}(r_1^2 - r^2). \qquad (2)$$

This equation gives the radial stress at any distance r from the axis of rotation.

Now considering the equilibrium of any particle we have by the same reasoning employed in Art. 80 the same two equations,

$$H + R = C \qquad (3)$$

and

$$rdR + Rdr = Hdr. \qquad (4)$$

Substituting in (4) the values of R and dR from (1) and (2) we have

$$r \cdot \frac{4\pi^2 n^2 w r dr}{g} + \frac{4\pi^2 n^2 w}{2g}(r_1^2 - r^2)\,dr = Hdr.$$

Integrating, the constant of integration being zero when $r = 0$ we have

$$H = \frac{4\pi^2 n^2 w}{2g}(r_1^2 + r^2). \qquad (5)$$

Equations (2) and (5) give the radial and hoop tensile stress respectively for any point at a distance r from the axis of rotation, and both are independent of r_2. Therefore, these equations can be applied to solid wheels, such as a grindstone, as well as to fly-wheels. As in the preceding article these formulæ have been deduced without considering the lateral deformation of Art. 79. The *true* hoop tension, taking this lateral deformation into consideration, would be $H_1 = H - kR$ and the *true* radial stress $R_1 = R - kH$. The value of the radial stress is greatest where $r = r^2$ (equation (2)), and for this value of r we have

$$R_1 = \frac{w \,.\, 2\pi^2 n^2}{g} (r_1^2 - kr_1^2 - r_2^2 - kr_2^2). \qquad (6)$$

The hoop tension is greatest where $r = r_1$, hence

$$H_1 = \frac{w \,.\, 2\pi^2 n^2}{g} \,.\, r_1^2. \qquad (7)$$

From equations (6) and (7) it is clear that the hoop tension, being the greater stress, will be the cause of rupture and also that the rupture will begin at the outside surface of the rim.

83. Bending Due to Centrifugal Force.—We will have a bending moment due to centrifugal force if both ends of a straight rod are constrained to move in circles, for example, the horizontal rod between the two driving wheels of a locomotive. In this case each point of the rod at any instant is moving in a circle and the centrifugal force due to this motion acts in the direction of the center of that circle for that instant. The *stress* caused acts in a diametrically opposite direction. Obviously the effect will be greatest when the horizontal rod is at its lowest point for here the weight of the rod itself acts down, as does also the stress due to centrifugal force. Calling the radius of the circle described by the ends of the rod r, and w the weight per unit length of the rod (considered of uniform cross-section) the radial stress due to centrifugal force will be

$$R = \frac{wv^2}{gr}.$$

The value of the linear velocity v is obtained from our knowledge of the speed of the engine, radius of the drive wheels, etc. For example, if the speed of the engine is S, radius of the drive wheels r_1, and radius of motion of the ends of the rod r, the linear velocity of any point in the horizontal rod will be $S\,\frac{r}{r_1}$, and this gives

$$R = \frac{wS^2 r}{gr_1^2}.$$

This value of R is uniform throughout the length of the rod, therefore, the rod may be considered as a beam loaded uniformly with R per unit length, and will have in addition when at its lowest position a uniform load due to its own weight. The stress at any point can then be readily found from the formula for bending $\frac{p}{y} = \frac{M}{I}$.

A connecting rod offers an important example of bending stress due to centrifugal force, but in this case one end of the rod is constrained to move in a straight line while the other end moves in a circle. At the instant the connecting rod is at right angles with the crank arm (obviously the greatest effect is produced when the load is perpendicular to the connecting rod) if the end of the connecting rod is above the center of motion of the crank arm, the weight of the connecting rod will then act down, while the line of action of the stress due to the centrifugal force will act upward in a direction parallel to the crank arm. Therefore, as before, the greater stress will be produced when the end of the connecting rod is *below* the center of motion of the crank arm. In this position, however, the force acting on the piston will put the connecting rod in *tension* so that it can readily sustain this bending stress. When the end of the connecting rod is above the center of motion of the crank arm, the stress produced by the pressure on the piston is compressive and the connecting rod is in the condition of a strut and any bending due to centrifugal force will become an important matter (Chapter XI). Taking the position then as shown in Fig. 71. The radius of the circle described by any point of the connecting rod will vary directly as the distance from A of the point, therefore, as the radial stress at any point due to the motion of the connecting rod is equal to $\frac{wv^2}{gr}$, where r is the radius of the circle described by the point and the value of v

depends upon constants and the length of r, we must have the load on the connecting rod due to the centrifugal force vary uniformly from zero at A to a maximum at B where it is equal to $\frac{wv^2}{ga}$ where a is the length of the crank arm. The bending moment due to this load (Art. 42), taking A for the origin, is

$$M_1 = \frac{wv^2}{6ga}\left(lx - \frac{x^3}{l}\right),$$

l being the length of the connecting rod, and x the distance

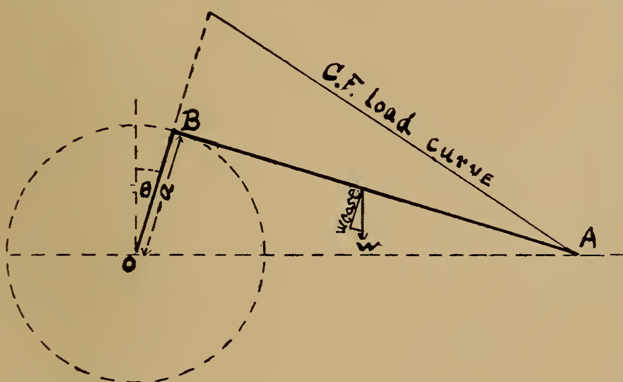

FIG. 71.

along it from A to any point. This bending moment tends to bend the rod so that the middle of it will move upward. The weight of the rod itself acts vertically downward and the component of the weight which acts perpendicular to the rod will be $w \cos \theta$ per unit length. This load causes a bending moment equal to (Art. 38)

$$M_2 = \frac{w \cos \theta}{2}(lx - x^2),$$

the effect of which is to cause the middle of the rod to move

downward. The *total* bending moment at any point distant x from A is then

$$M = M_1 - M_2 = \frac{wv^2}{6ga}\left(lx - \frac{x^3}{l}\right) - \frac{w \cos \theta}{2}\,(lx - x^2).$$

Having the bending moment at any point due to the motion and weight of the rod the stress due to these causes is readily found from $\frac{p}{y} = \frac{M}{I}$. But it may be repeated that a connecting rod in addition to the above stress suffers a compressive stress, when in this position, due to the pressure on the piston and also a bending stress due to this compressive load which puts the rod in the condition of the column or strut discussed in Chapter XI. Referring to that chapter it will be seen that the comparatively slight bending due to centrifugal force assumes important dimensions when we consider that the least variation of the axis of the connecting rod from the line of action of this large compressive load makes the opportunity for the load to cause bending.

84. Flat Plates.—Experiment has proved that a circular flat plate when subjected to too great a pressure on one side always breaks along a diameter. Any diameter, then, of a circular plate is perpendicular to the greatest tensile stress due to bending caused by the pressure on one side of the plate. Let us consider a circular plate simply supported at the circumference and subjected to a uniformly distributed load on the side opposite to the support. If the plate were *fixed* at the circumference it would be stronger than the one we are considering, just as a beam fixed at the ends is capable of supporting a greater load than if it were free at the ends. We will consider our support as a ring on which the plate rests, the pressure being uniformly distributed on the upper side.

Let Fig. 72 represent such a plate, the support being at the

circumference, and the load w lbs. per unit area. We must first find the bending moment at the diameter AB. The area of the triangular element shown is $\frac{r}{2}. rd\theta$, and the load on it is $w\frac{r^2}{2}d\theta$. The supporting force under the arc $rd\theta$ must be equal to this load, and its distance from the diameter AB is $r\cos\theta$. The center of gravity of the uniform load on

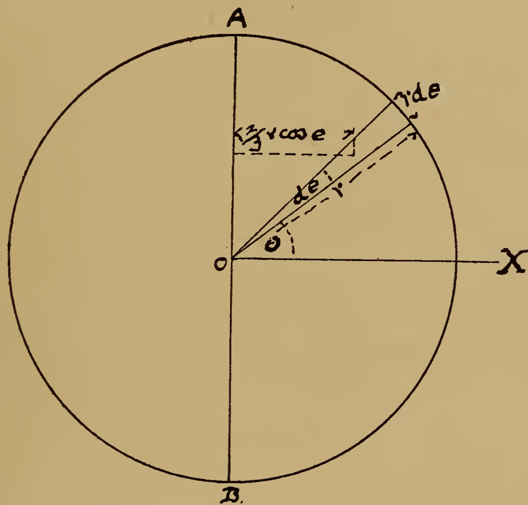

FIG. 72.

the triangular element is at a distance $\frac{2}{3}r$ from the center and $\frac{2}{3}r\cos\theta$ from the diameter AB. The bending moment at the diameter AB due to these two forces is then

$$dM = -\frac{wr^2}{2}\,d\theta\,.\,\tfrac{2}{3}r\cos\theta + \frac{w}{2}\,r^2 d\theta\,.\,r\cos\theta.$$

Integrating between the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for θ

$$M = \frac{wr^3}{3},$$

which is the bending moment due to the loads on one side of the diameter AB. Letting t be the thickness of the plate the moment of inertia of the section through AB about the neutral axis is $I = \frac{rt^3}{6}$. We have then from the equation of bending $\frac{p}{y} = \frac{M}{I}$. ($y$ in this equation being equal to $\frac{t}{2}$.)

$$p = \frac{wr^3}{3} \cdot \frac{t}{2} \cdot \frac{6}{rt^3} = w\frac{r^2}{t^2},$$

which gives the stress on any section through a diameter. In designing a plate of this kind we would know the pressure to which it would be subjected and the dimensions of the opening it would cover, so if we put the limiting tensile strength of the material we are to use for p we can solve the above equation for t, the thickness necessary.

If the load instead of being uniform is a concentrated one; calling it W and supposing it is to be at the most effective position, the center, and to bear uniformly on a small circle of radius r_1 concentric with the supporting ring, the part of the load on one side of any diameter will be $\frac{W}{2}$ and its center of gravity will be at a distance $\frac{4r_1}{3\pi}$ from the diameter. The supporting force will be uniform throughout the length of the semicircle of the ring on the same side of the diameter as this load, and the center of gravity of this semicircle is at a distance $\frac{2r}{\pi}$ from the diameter and the total supporting force on this semicircle is obviously $\frac{W}{2}$. The bending moment at this diameter will be

$$M = \frac{W}{2} \cdot \frac{2r}{\pi} - \frac{W}{2} \cdot \frac{4r_1}{3\pi} = \frac{W}{\pi}\left(r - \frac{2r_1}{3}\right),$$

t being the thickness of the plate and the moment of inertia

of the section be $I = \frac{rt^3}{6}$ as before we have for the stress in the section

$$p = \frac{My}{I} = \frac{W}{\pi}\left(r - \frac{2r_1}{3}\right) \cdot \frac{t}{2} \cdot \frac{6}{rt^3} = \frac{3W}{\pi t^2}\left(1 - \frac{2r_1}{3r}\right).$$

If $r_1 = r$; or, which is the same thing, if the plate bears a uniform load, we have, remembering W will now be equal to $\pi r^2 w$,

$$p = w\frac{r^2}{t^2},$$

and if $r_1 = 0$, or the load is concentrated at a point,

$$p = \frac{3W}{\pi t^2}.$$

The stress given by these formulæ is the maximum stress which occurs as one would expect at the *center* of the plate. Experiment proves this to be the case, for such plates begin to crack at the center and the crack extends from there, along a diameter, to the edge of the plate. Experiment also proves the above formulæ to give very approximate results, so, though this may be a tentative method of deducing them (the assumptions not being strictly true), they will serve for all practical purposes.

Rectangular Plates.—Experiment shows that rectangular plates when the length is not more than about twice the width, will crack along a diagonal. Therefore, if the sides are a and b (Fig. 73), the diagonal section will support the greatest bending stress. Suppose the plate of thickness t to support a uniform load of w per unit area. The load on one side of the diagonal will be $\frac{wab}{2}$ and its center of gravity will be at a distance $\frac{x}{3}$ from the diagonal. Assuming the supporting force to be uniform along the edges, the part acting along the edge a will act at its center of gravity which is at

a distance $\frac{x}{2}$ from the diagonal, as is also the center of gravity of the part acting along the edge b. These two edges will support the whole load on this side of the diagonal, or the whole supporting force here will equal $\frac{wab}{2}$. The bending moment about the diagonal then is

$$M = -\frac{wab}{2} \cdot \frac{x}{3} + \frac{wab}{2} \cdot \frac{x}{2} = \frac{wabx}{12}.$$

Calling g the length of the diagonal, the moment of inertia

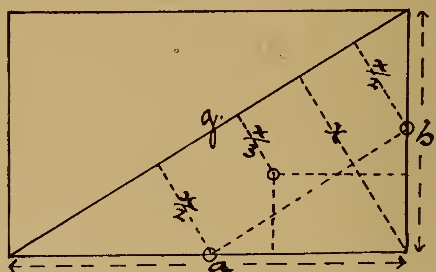

FIG. 73.

of the section through it about its neutral axis is, $I = \frac{gt^3}{12}$ and $y = \frac{t}{2}$. The equation of bending gives us

$$p = \frac{My}{I} = \frac{wabx}{12} \cdot \frac{t}{2} \cdot \frac{12}{gt^3} = \frac{wabx}{2gt^2}.$$

Referring to the figure, $g = \sqrt{a_2 + b_2}$ and from similar triangles $x = \frac{ba}{\sqrt{a^2 + b^2}}$, substituting these values

$$p = \frac{wa^2b^2}{2(a^2 + b^2)t^2}.$$

If the plate is square $a = b$ and we have

$$p = \frac{wa^2}{4t^2}.$$

These formulæ should for practical purposes have a factor on the right member of the equation of 1.5 if the support is a *fixed* one (riveted joint), and of 5 if there is only a simple support.

The formula for elliptical plates which experiment shows break along the major axis is for *wrought* iron and *steel* plates simply suported around the edges,

$$p = \tfrac{8}{3} \cdot \frac{wa^2b^2}{t^2(a^2 + b^2)}$$

(if *cast* iron is used the coefficient is 3 instead of $\frac{8}{3}$) where w is the load per unit area, t the thickness, and a and b the semi-major and semi-minor axes respectively. The theoretical solution for elliptical plates is very difficult because it involves elliptical integrals and because the supporting force is not uniform around the edge, as indeed is also the case for rectangular plates, but the variation for the latter is not excessive unless the plate is more than two or three times as long as it is wide. In fact all the above deductions are approximations for the assumptions made are not strictly true. The formulæ, however, agree closely with the results obtained experimentally and may be assumed correct for all practical purposes.

The subject of flat plates is probably the most unsatisfactory one in the study of strength of material and practical engineers have different methods for each form of plate. Fault may be found with the assumptions of most of these methods, though they all give approximate results.

85. The principle of least work may be stated as follows: For stable equilibrium the stresses in any structure must have such values that the potential energy of the system is a minimum. The stresses of course must be within the elastic limit of the material. When forces act upon bodies which conform

to Hook's law, the principle of least work may be applied to determine some unknown reaction. Perhaps the best-known example is that of a four-legged table which supports an unsymmetrically placed load. We can get from our knowledge of statics three equations to find the part of the load supported by each leg, and the solution can be arrived at by means of the fourth condition furnished by the principle of least work.

Distribution of Stress.—We have all along assumed that the stress is uniform throughout a section. As a matter of fact this is not the case. It can be mathematically proved that the shearing stress in a rod of square section varies as the ordinates of a parabola, being zero where the normal stress due to bending is a maximum, and a maximum where the normal stress due to bending is zero (at the neutral surface). Again, the shear parallel to the neutral axis in a rod of square section is zero, but in a rod of circular section it has a finite value.

Velocity of Stress.—When a force is applied to a piece of material the stress is not instantaneously produced, but moves with a wave motion through the mass. The velocity of this motion can be found and it is shown to depend upon the stiffness and density of the material. The velocity of stress should be taken into account in problems involving impact and suddenly applied loads.

Internal Friction.—When material is subjected to force and deformation occurs the molecules of the material move and this motion is resisted by internal friction. Heat is produced and for the time between the application of the load and the complete rearrangement of the molecules, the stresses at planes through the material are changing; for this time then our formulæ for planes of maximum stress are not correct.

Fatigue of Materials.—Experiment proves that material will break if subjected to *repeated* stress even if the stress be somewhat below the ultimate strength of the material. Experiment also shows that the greater the number of applications the less becomes the value of the stress necessary to cause rupture. For example, about a hundred thousand applications of a stress of 49,000 lbs. to wrought iron will cause rupture, but if 500,000 applications were made the stress need be only 39,000 lbs. The loss of strength due to repeated stress is known as the *Fatigue of Materials.* This fatigue is more marked if the stress alternates from tension to compression and back again.

The preceding facts have been mentioned to give the student the knowledge of their existence so that if inclined he may look them up in more complete works on the subject. The time limit of the course for which this book has been written precludes the possibility of entering into discussion of many subjects of importance, such as the stress in hooks, links, springs, rollers, foundations, arches and many others.

Miscellaneous Examples:

1. A reinforced concrete beam is 48 ins. wide, 54 ins. long and 5 ins. deep. It has a sectional area of 3.6 sq. in. of steel, the center of gravity of which is $\frac{9}{2}$ in. below the top of the beam which is uniformly loaded with 2400 lbs. per in. length. (Use $\frac{E_s}{E_c} = 10.$) Find the position of the neutral surface, and the stress in the steel.

Ans. Neutral surface 3″.45 below top.

2. A reinforced concrete beam is 60 ins. long, 48 ins. wide and 5 ins. deep. The sectional area of the steel is 2.4 sq. in. and its center of gravity is $\frac{9}{2}$ in. below the top of the beam. The load is uniform and equals 3600 lbs. per in. length.

Find the position of the neutral surface and the stress in the concrete and steel. $\frac{E_s}{E_c} = 10.$

Ans. Neutral surface 1″.68 from top.

3. If the allowable unit stress in concrete be 500 lbs. per sq. in., and that for steel be 10,000 lbs. per sq. in., what percentage of steel must be put in a beam 20 ins. wide by 10 ins. deep if the steel is at a distance of 9 ins. from the top?

Ans. 0.75%.

4. If the allowable unit stress for concrete and steel are 500 and 25,000 lbs. per sq. in. respectively, what is the resisting moment of a beam 7 ins. wide and 10 ins. deep if the reinforcement of steel is 1% of the sectional area and placed at the lowest point of the beam?

Ans. 92,800 in.-lbs.

5. A beam is 8 ft. long and 1 ft. wide. The concrete is 5 ins. thick and the steel reinforcement is 1 in. from the bottom of the beam. What area of steel section will be necessary, and using $\frac{1}{2}$ in. square bars what should be the distance between them if a floor is made in this way?

6. The cross-section of a steel bar is 16 sq. in. The bar is put under a stress of 27,000 lbs. per sq. in. If $k = \frac{1}{3}$ what is the sectional area while under stress?

Ans. 15.99 sq. in.

7. A steel bar is 2.5 in. in diameter and 18.5 ft. long. What are its length and sectional area under a pull of 64,000 lbs.?

8. The inside of a cylinder is 6 ins., its outside radius is 1 ft. The internal pressure is 600 lbs. per sq. in., and the external pressure is that due to the atmosphere. What are the hoop and radial stresses at the outside and inside surfaces, also midway between these surfaces?

Ans. Hoop stress inside, 960; outside, 375 lbs. per sq. in.

9. A solid cylinder is under a uniform external pressure of 14,000 lbs. per sq. in. Show that the hoop and radial stresses are uniform and give values.

10. A gun is built of a tube, inside radius 3 ins., outside radius 5 ins., and a jacket 2 ins. thick. Before assembling, the difference between the outside radius of the tube and the inside radius of the jacket was .004 in. What are the stresses at the outside and inside surfaces?

Ans. Hoop compression, 14,400 lbs. per sq. in.

11. In example 10 what powder pressure when firing the gun will just reverse the stress at the inner surface of the tube?

12. The radii of a gun composed of tube and jacket are $r_3 = 3''.04$, $r_2 = 5''.8$, and $r_1 = 9''.75$. The allowable unit stress is 50,000 lbs. per sq. in. for both tension and compression. What are the radii before assembling?

Ans. Radius of bore, $3''.0451$; outside radius of tube, $5''.805$; inner radius of jacket, $5''.7915$; outer radius, $9''.7436$ ins.

13. What internal powder pressure will stress the gun of example 12 to just 50,000 lbs. per sq. in. tension?

Ans. 51,100 lbs. per sq. in.

14. What is the greatest tangential stress in a cast-iron fly-wheel 30 ft. in diameter, rim 1 in. thick and 4 ins. wide, when it is making 60 revolutions per minute?

Ans. Very roughly, 800 lbs. per sq. in.

15. What is the stress in a cast-iron fly-wheel rim having a linear velocity of 1 mile a minute?

Ans. Roughly, 750 lbs. per sq. in.

16. What diameter should a fly-wheel have if it is to make 100 revolutions per minute, and the maximum allowable linear velocity is 6000 ft. per min.?

Ans. Roughly, 19 ft.

17. A cast-iron bar 9 ft. long, 3 ins. wide and 2 ins. thick revolves about an axis $\frac{3}{4}$ in. in diameter through it at a distance of $4\frac{1}{2}$ ft. from one end. How many revolutions per second will produce rupture?

18. A solid steel circular saw is 4 ft. in diameter, and makes 2700 revolutions per minute. What is the stress at the circumference? How many revolutions per minute will cause a stress of 35,000 lbs.?

19. An engine is making 750 revolutions per min., the connecting rod is 2 ft. long, and the crank arm 6 ins., material steel. What is the bending stress, due to centrifugal force, on the connecting rod if the area of its section be 1.5 sq. ins.?

20. A cast-iron fly-wheel, mean diameter of rim 20 ft., makes 90 revolutions per minute. The cross-section of the rim is 10 sq. ins. What is the stress?

21. What must be the thickness of a cast-iron cylinder head 36 ins. in diameter, allowable stress 3600 lbs. per sq. in., to sustain a load of 250 lbs. per sq. in.?

Ans. 5 ins.

22. The allowable stress for steel being 12,000 lbs. per sq. in., how thick would a steel head for the cylinder of example 21 have to be?

Ans. 2.6 ins.

23. A circular steel plate is 24 ins. in diameter and 1.5 ins. thick. It carries a load of 4000 lbs. at the center resting on a circle of 1 in. diameter. What is the maximum stress?

Ans. 35,500 lbs. per sq. in.

24. Suppose the load of example 23 were distributed on a surface of 3 ins. diameter. What would be the stress?

Ans. 10,100 lbs. per sq. in.

25. What must be the thickness of a steel plate 4 ft. square to carry 200 lbs. per sq. ft.

Ans. 0.2 ins.

26. What uniform load can be carried by a wrought-iron plate $\frac{3}{8}$ in. thick, 5 ft. long and 3 ft. wide?

Ans. About 12 lbs. per sq. in.

27. A floor 18 ft. long, 15 ft. wide is made of concrete 4 ins. thick, with 1 in. square wrought-iron rods, spaced 1 ft. apart and at .75 in. from the bottom of the concrete. The floor carries a load of 150 lbs. per sq. ft. What is the maximum stress? $\frac{E_s}{E_c} = 15$.

Ans. Stress is about 450 lbs. per sq. in.

28. An elliptical plate (cast iron) has a major axis 24 ins. long, a minor axis 16 ins. and is under a uniform pressure of 22 lbs. per sq. in. The allowable stress is 3000 lbs. per sq. in., and the plate is simply supported at the edges. What must be the thickness?

Ans. 1 in.